Centro de Referência em Radiocomunicações

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Instantaneous Spectral Analysis (ISA)

Henry Douglas Rodrigues – October 17th 2016

ISA definition

A new spectral analysis tool which generates a non-stationary frequency domain.

Each spectral component presents a continuously-varying amplitude over the transformation time.

The underlying math is based on the generalization of Euler's formula*.

$$e^{ti} = \cos(t) + i\sin(t)$$

(The foundation formula of the telecommunications industry)

*J. Prothero, "*Euler's Formula for Fractional Powers of i*", 2007 [Online]. Available: http://astrapi-corp.com/wp-content/uploads/2014/01/Eulers_Formula_for_Fractional_Powers_of_i.pdf

Generalization of Euler's Formula

For fractional powers of i: $e^{ti^{2^{(2-m)}}}$

m	0	1	2	3	4	5	$\cdots$
$e^{ti^{2^{(2-m)}}}$	e^{t}	e^{-t}	e^{it}	$e^{t\sqrt{i}}$	$e^{t\sqrt[4]{i}}$	$e^{t\sqrt[8]{i}}$	$\cdots$

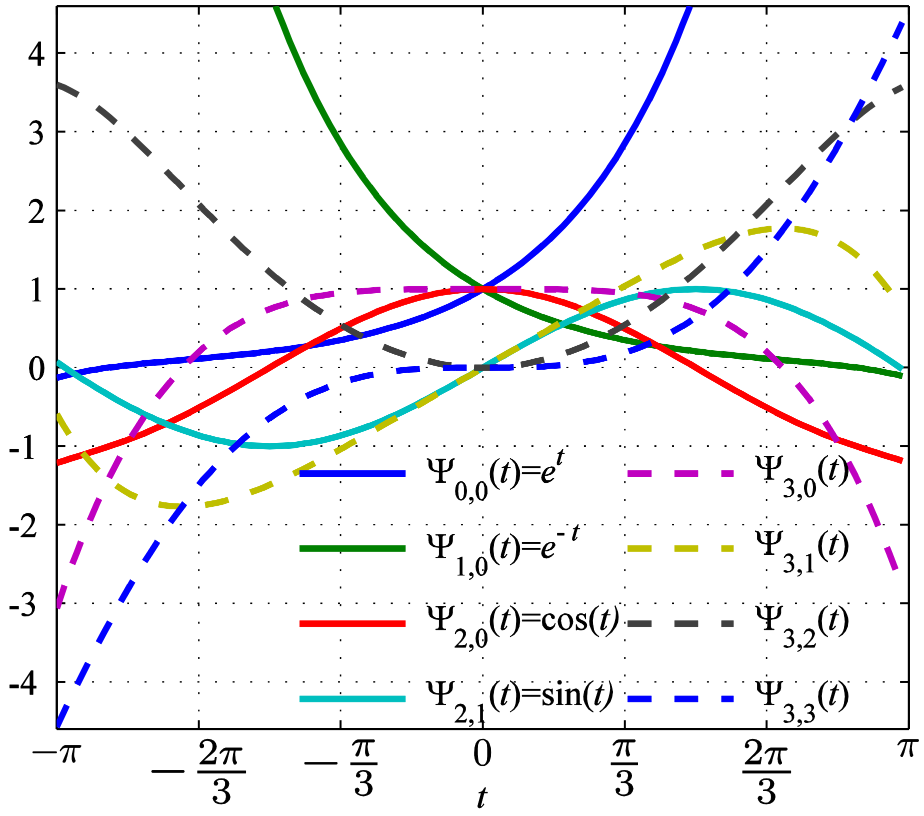
Expanding as a Taylor series:

$$e^{ti^{(2^{2-m})}} = \sum_{n=0}^{\lceil 2^{m-1}\rceil -1} i^{n2^{2-m}} \psi_{m,n}(t)$$

where

$$\psi_{m,n}(t) = \sum_{q=0}^{\infty} (-1)^{q\lceil 2^{1-m}\rceil} \frac{t^{q\lceil 2^{m-1}\rceil + n}}{(q\lceil 2^{m-1}\rceil + n)!}$$

are the *Cairns series* functions.



ISA Procedure; Step 1/4

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Any sequence $\mathbf{x} = [x_0,\, x_1,\, \ldots,\, x_{K-1}]^{\mathrm{T}}$ with $\mathbf{x} \in \mathbb{C}^{K \times 1}$ can be represented by a Taylor polynomial.

Fit a Taylor polynomial that represents the sequence

$$\mathbf{h} = \left(\mathbf{B}^{\mathrm{H}}\mathbf{B}\right)^{-1} \mathbf{B}^{\mathrm{H}}\mathbf{x}$$

where $\mathbf{B} = [\mathbf{b}_0,\, \mathbf{b}_1,\, \ldots,\, \mathbf{b}_\gamma,\, \ldots,\, \mathbf{b}_{K-1}]$ with $\mathbf{b}_\gamma = \left[\frac{t_0^\gamma}{\gamma!},\, \frac{t_1^\gamma}{\gamma!},\, \ldots,\, \frac{t_{K-1}^\gamma}{\gamma!}\right]^{\mathrm{T}}$,

$t_k = \frac{2\pi}{K-1}k - \pi$, and $k = 0,\, 1,\, \ldots,\, K-1$.

ISA Procedure; Step 2/4

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The approximation of the Cairns series function is given by

$$\boldsymbol{\Psi} = \mathbf{B}\mathbf{C}$$

The sequence can also be represented by the linear combination

$$\boldsymbol{\Psi}\mathbf{c} = \mathbf{x}$$

Project the polynomial into the Cairns series functions.

$$\mathbf{B}\mathbf{C}\mathbf{c} = \mathbf{B}\mathbf{h}$$

$$\mathbf{c} = \mathbf{C}^{-1}\mathbf{h}$$

where $\mathbf{c} = [c_{0,0},\ c_{1,0},\ c_{2,0},\ c_{2,1},\ c_{3,0},\ c_{3,1},\ c_{3,2},\ c_{3,3},\ \cdots]^{\mathrm{T}}$

At this point any Taylor polynomial can be synthesized as

$$p(t) = \sum_{m=0}^{M} \sum_{n=0}^{\lceil 2^{m-1} \rceil - 1} c_{m,n}\psi_{m,n}(t)$$

	1	t	$\frac{t^2}{2!}$	$\frac{t^3}{3!}$	$\frac{t^4}{4!}$	$\frac{t^5}{5!}$	$\frac{t^6}{6!}$	$\frac{t^7}{7!}$	$\cdots$
$\psi_{0,0}(t) = e^t$	1	1	1	1	1	1	1	1	$\cdots$
$\psi_{1,0}(t) = e^{-t}$	1	-1	1	-1	1	-1	1	-1	$\cdots$
$\psi_{2,0}(t) = \cos(t)$	1	0	-1	0	1	0	-1	0	$\cdots$
$\psi_{2,1}(t) = \sin(t)$	0	1	0	-1	0	1	0	-1	$\cdots$
$\psi_{3,0}(t)$	1	0	0	0	-1	0	0	0	$\cdots$
$\psi_{3,1}(t)$	0	1	0	0	0	-1	0	0	$\cdots$
$\psi_{3,2}(t)$	0	0	1	0	0	0	-1	0	$\cdots$
$\psi_{3,3}(t)$	0	0	0	1	0	0	0	-1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

$\mathbf{C}^{\mathrm{T}} \in \mathbb{R}^{K \times K}$

ISA Procedure; Step 3/4

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Cosine and sine functions produced by sums of complex exponentials

$$\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}! + \cdots = \frac{1}{2}(e^{it} + e^{-it})$$

$$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots = \frac{1}{2i}(e^{it} - e^{-it})$$

This characteristic also holds for the generalized exponential description

$$E_{m,n}(t) = \frac{1}{\lceil 2^{m-1} \rceil} \sum_{p=0}^{\lceil 2^{m-1} \rceil - 1} i^{-n(2p+1)2^{2-m}} e^{ti^{(2p+1)2^{2-m}}}$$

Convert from the Cairns series functions to the Cairns exponential functions.

$$E_{m,n}(t) = \psi_{m,n}(t)$$

ISA Procedure; Step 4/4

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Considering $e^{ti^{(2^{2-m})}} = e^{t\cos(\pi 2^{1-m})} e^{it\sin(\pi 2^{1-m})}$

The Taylor polynomial can be described as

$$p(t) = \sum_{m=0}^{M} \sum_{n=0}^{\lceil 2^{m-1} \rceil - 1} \frac{c_{m,n}}{\lceil 2^{m-1} \rceil} \sum_{p=0}^{\lceil 2^{m-1} \rceil - 1} i^{-n(2p+1)2^{2-m}} e^{t\cos(\pi(2p+1)2^{1-m})} e^{it\sin(\pi(2p+1)2^{1-m})}$$

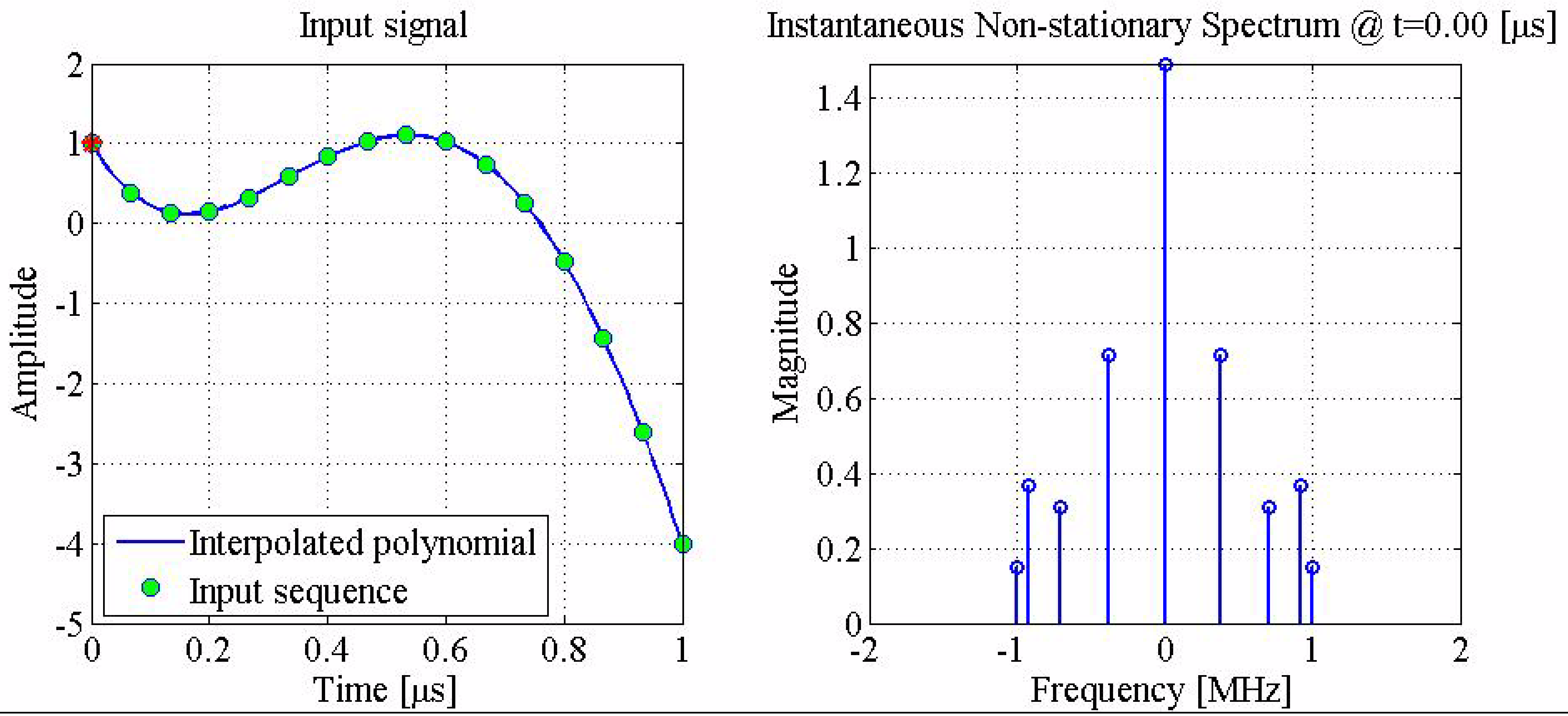
Constant
Real-valued exponential
Complex sinusoid

Obtain continuously time-varying amplitudes for each spectral component.

$$p(t) = \overbrace{\left[c_{0,0}e^{t} + c_{1,0}e^{-t}\right]}^{f=0\ (\mathrm{DC})} + \overbrace{\left[c_{2,0}\frac{1}{2} - c_{2,1}\frac{i}{2}\right] e^{it}}^{f=\frac{1}{2\pi}\ (\mathrm{maximum})} + \overbrace{\left[c_{2,0}\frac{1}{2} + c_{2,1}\frac{i}{2}\right] e^{-it}}^{f=-\frac{1}{2\pi}\ (\mathrm{minimum})} +$$

$$\underbrace{\sum_{m=3}^{M} \sum_{p=3\cdot 2^{m-3}}^{5\cdot 2^{m-3}-1} \left[\sum_{n=0}^{\lceil 2^{m-1} \rceil - 1} \frac{c_{m,n}}{\lceil 2^{m-1} \rceil} \left(\frac{e^{t\cos(\pi(2\langle p\rangle_{2^{m-1}}+1)2^{1-m})}}{i^{n(2\langle p\rangle_{2^{m-1}}+1)2^{2-m}}} + \frac{e^{-t\cos(\pi(2\langle p\rangle_{2^{m-1}}+1)2^{1-m})}}{i^{n(2(-p+3\cdot 2^{m-2}-1)+1)2^{2-m}}} \right) \right] e^{it\sin(\pi(2\langle p\rangle_{2^{m-1}}+1)2^{1-m})}}_{-\frac{1}{2\pi} < f < \frac{1}{2\pi}}$$

ISA Non-Stationary Spectrum



Comparison with Fourier Transform

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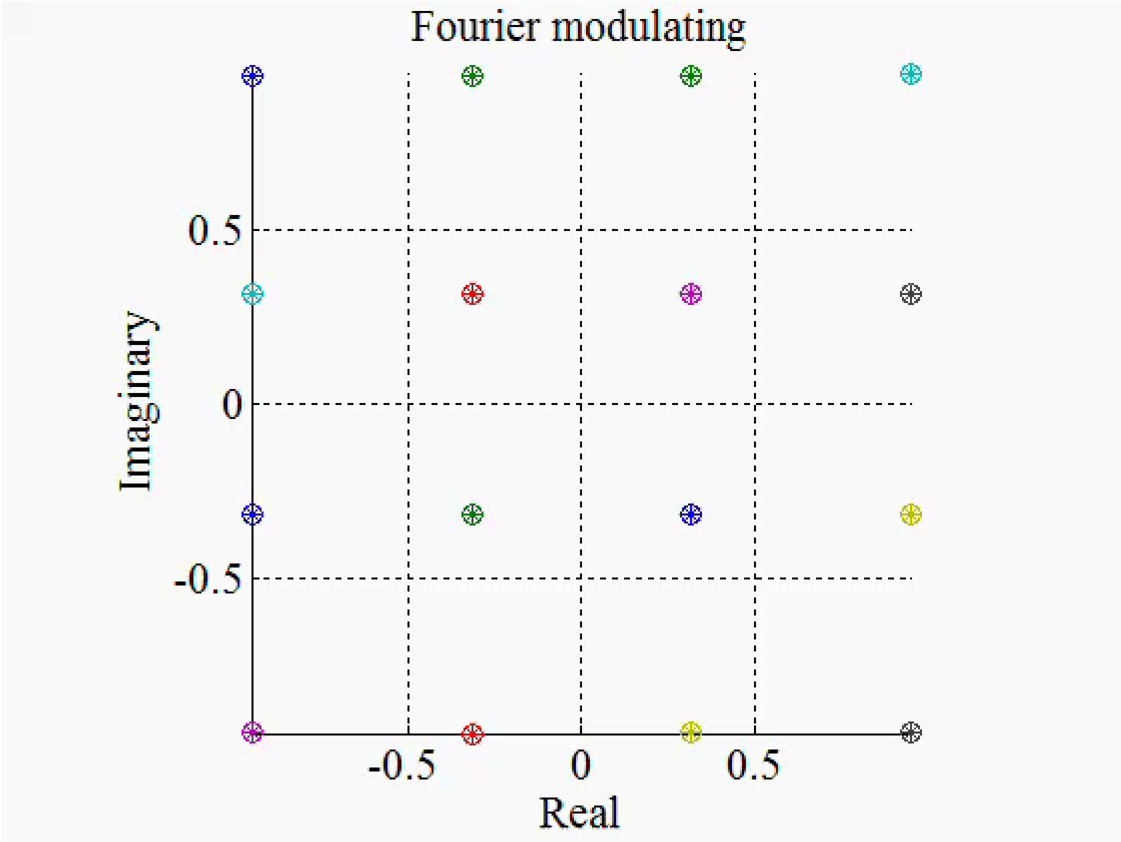
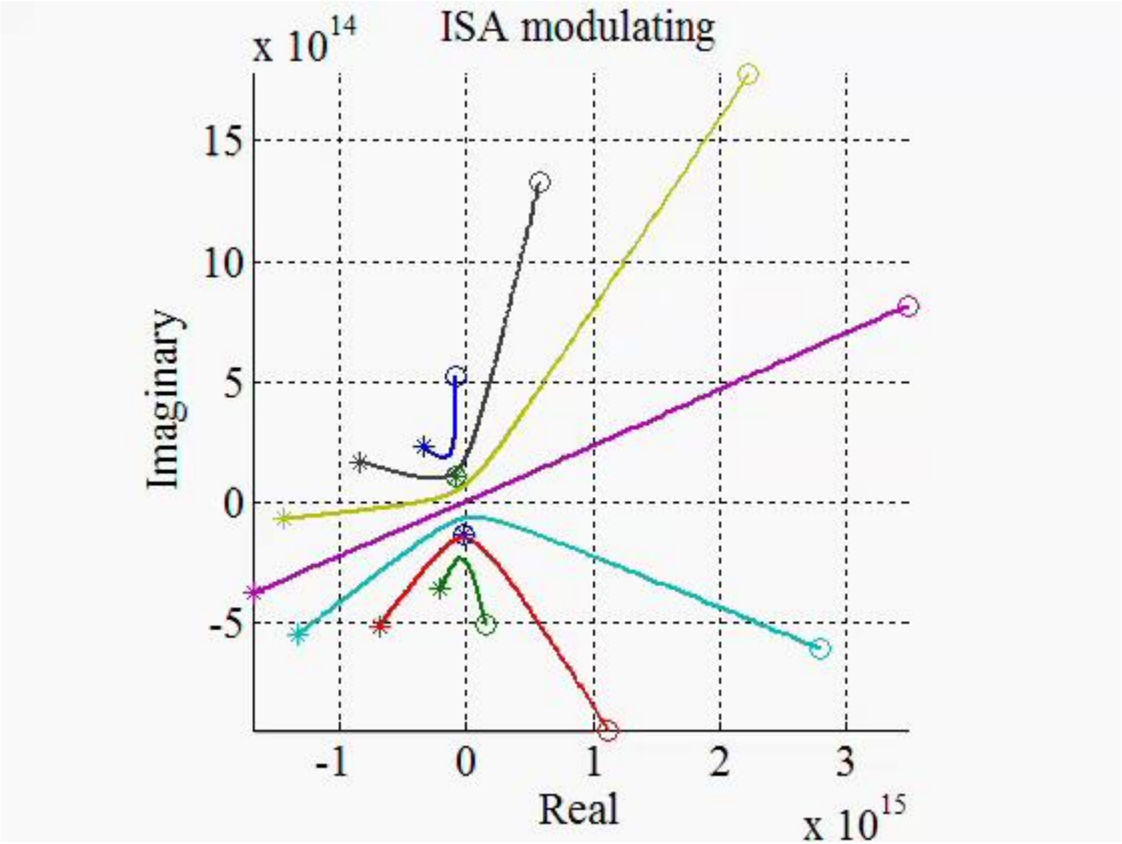
Non-stationarity





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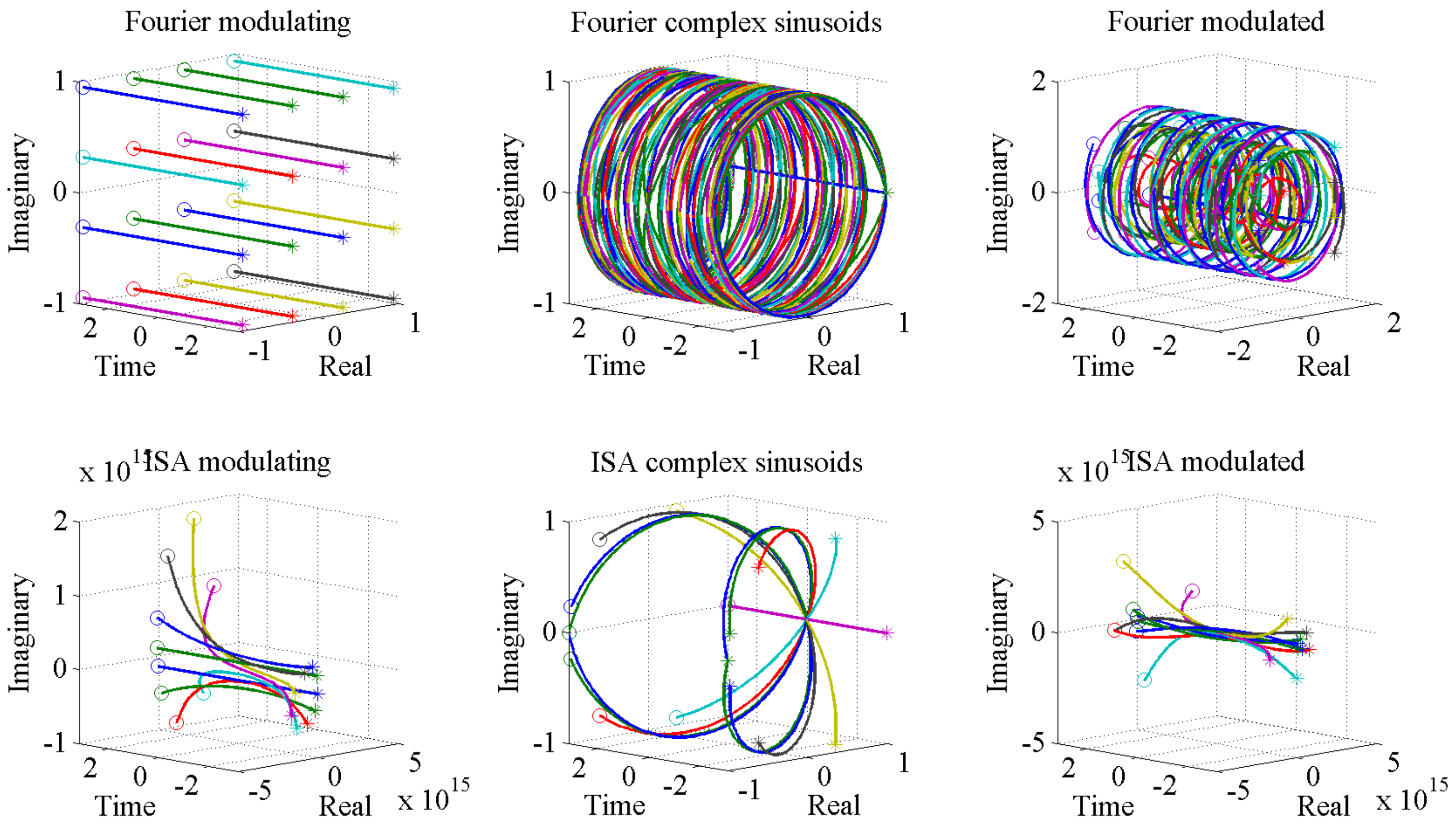
Graphical interpretation



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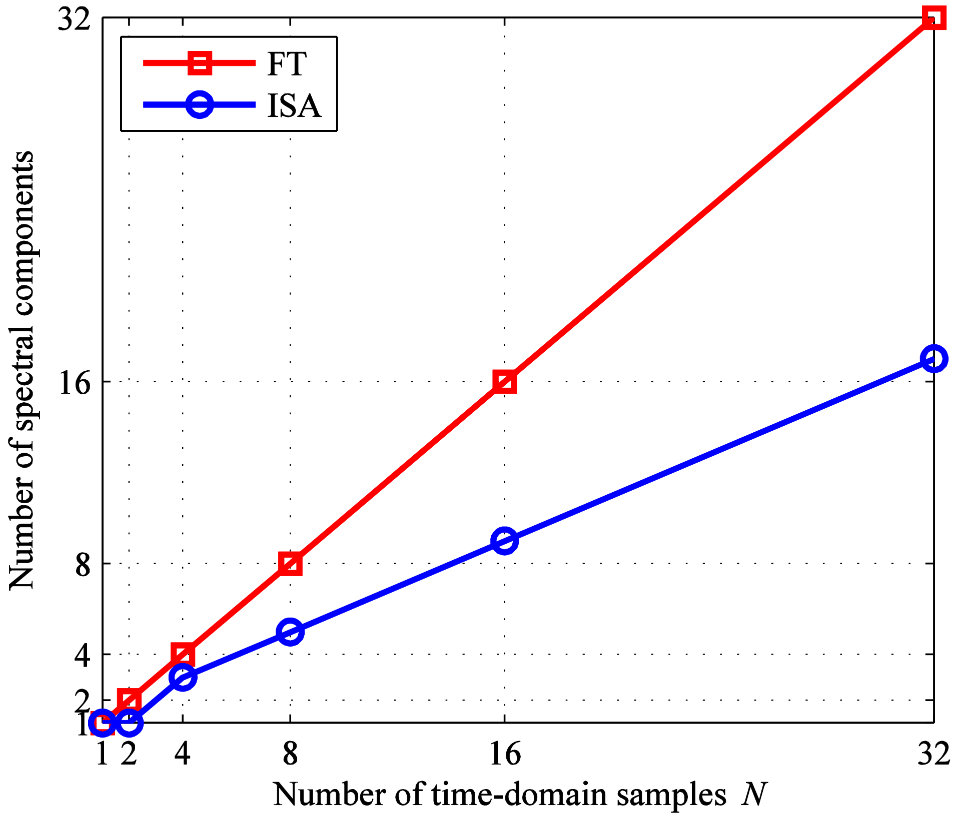
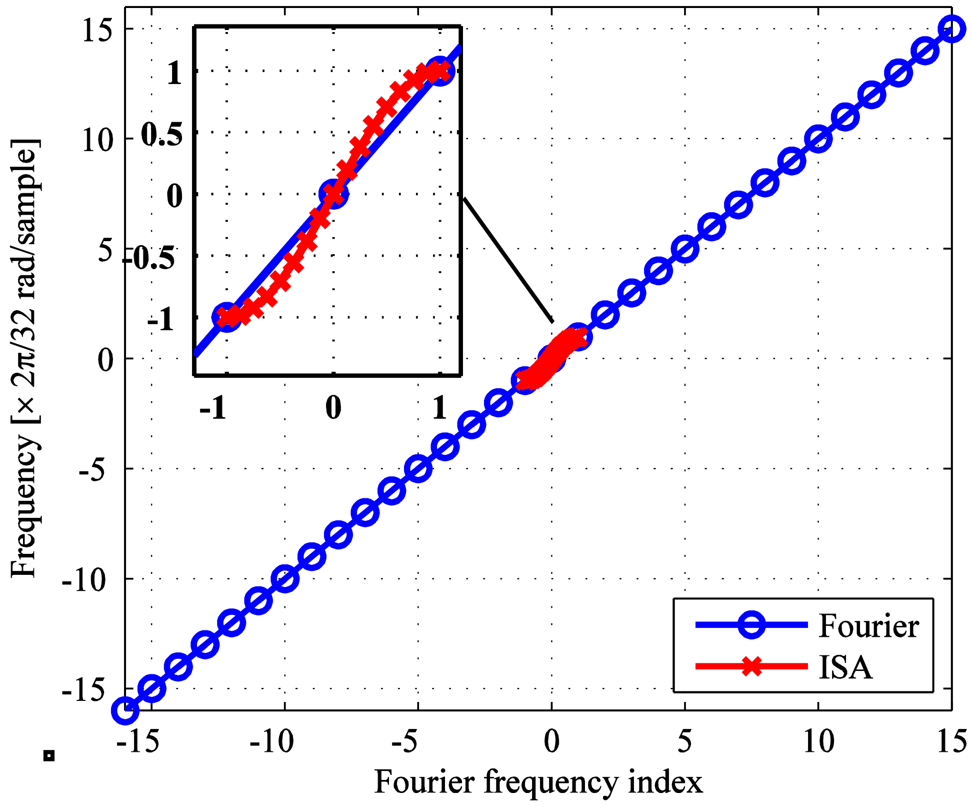
Number of spectral components



$$N_f = 1 + 2\sum_{m=2}^{M} \lceil 2^{m-3} \rceil$$

Spectral components spacing



$$f_{p,m} = \frac{\sin(\pi(2p+1)2^{(1-m)})}{2\pi}$$

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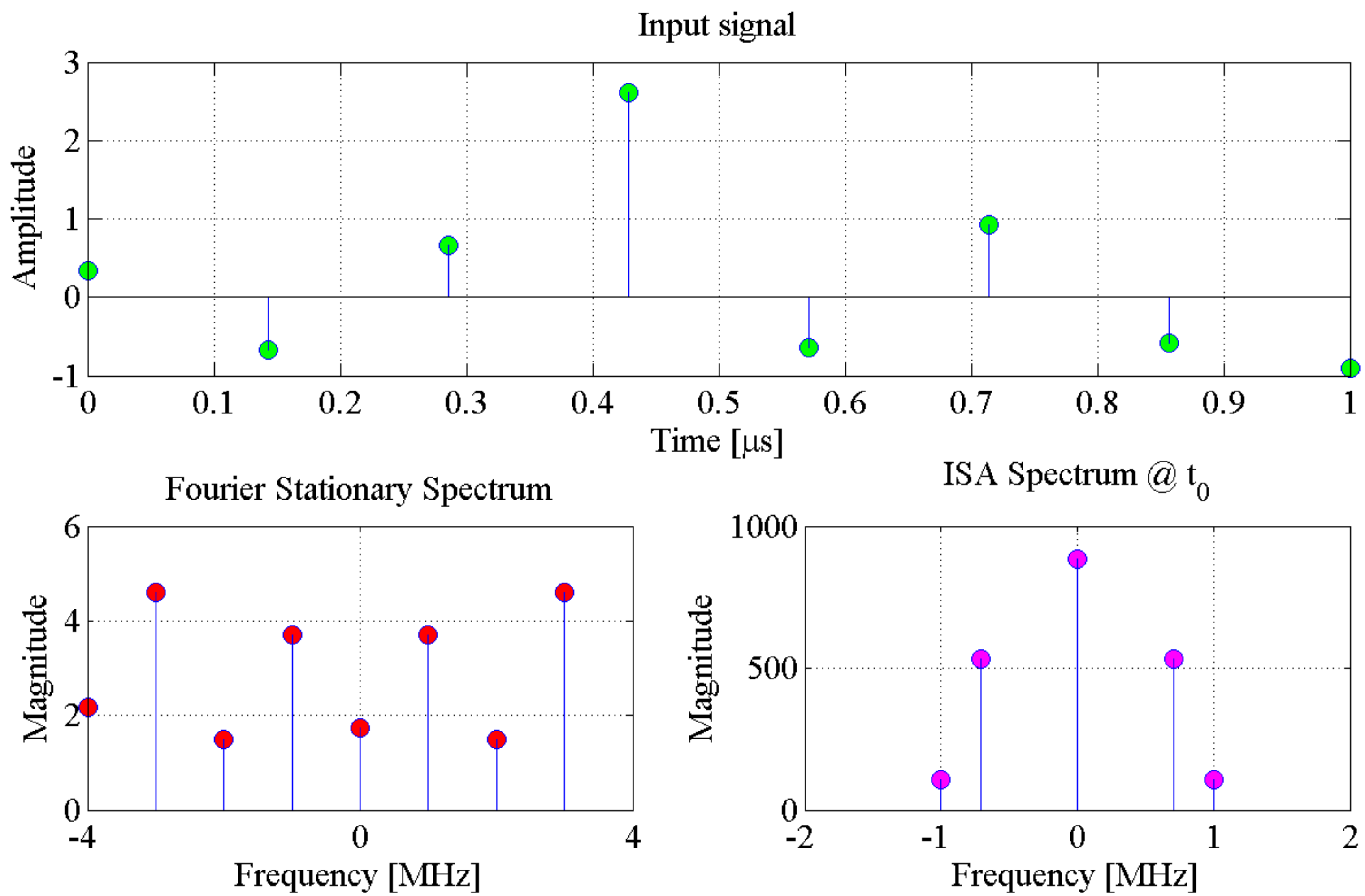
Basis functions Bandwidth Compression: 8 independent amplitude values within 1 μs.



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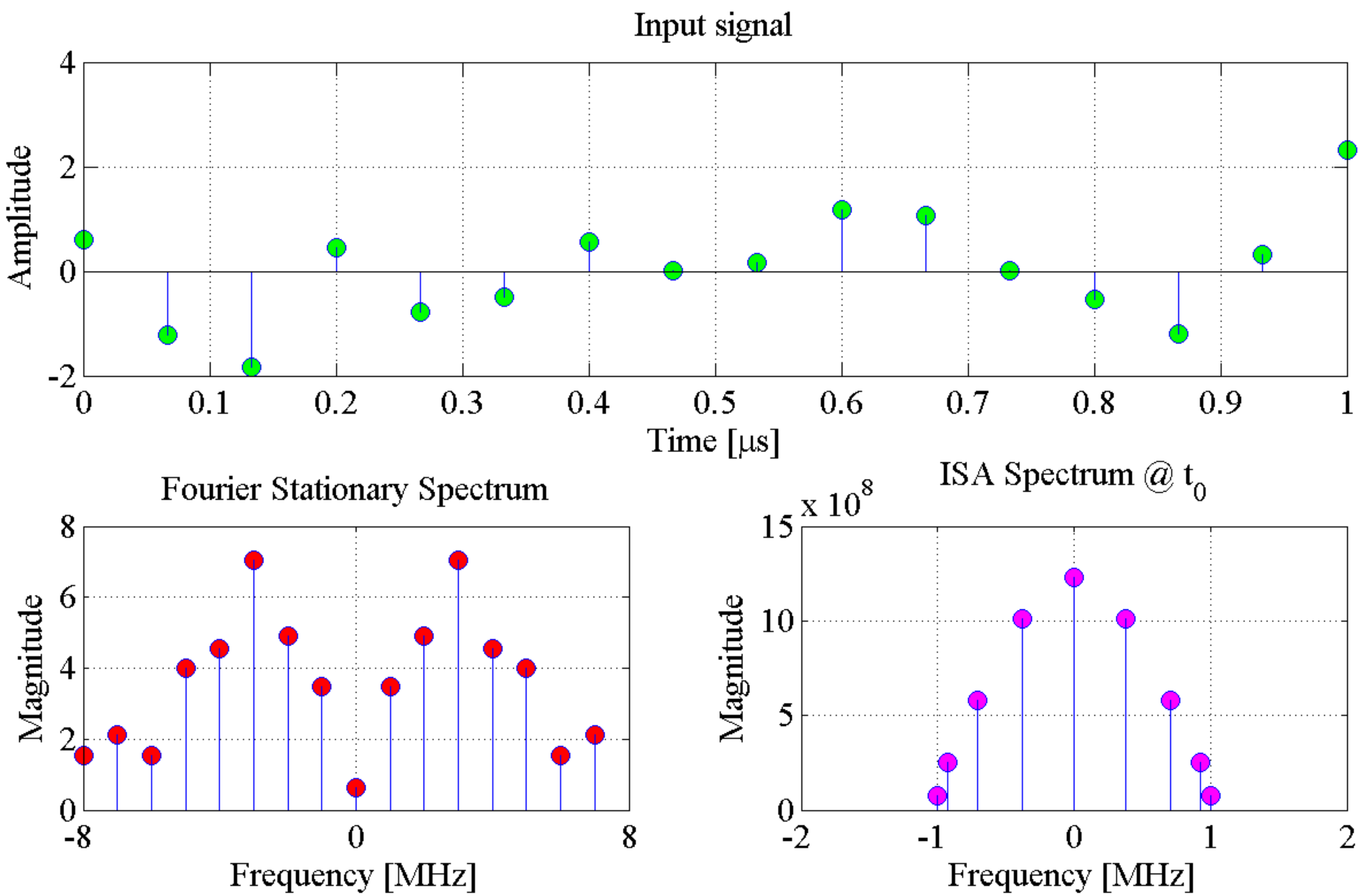
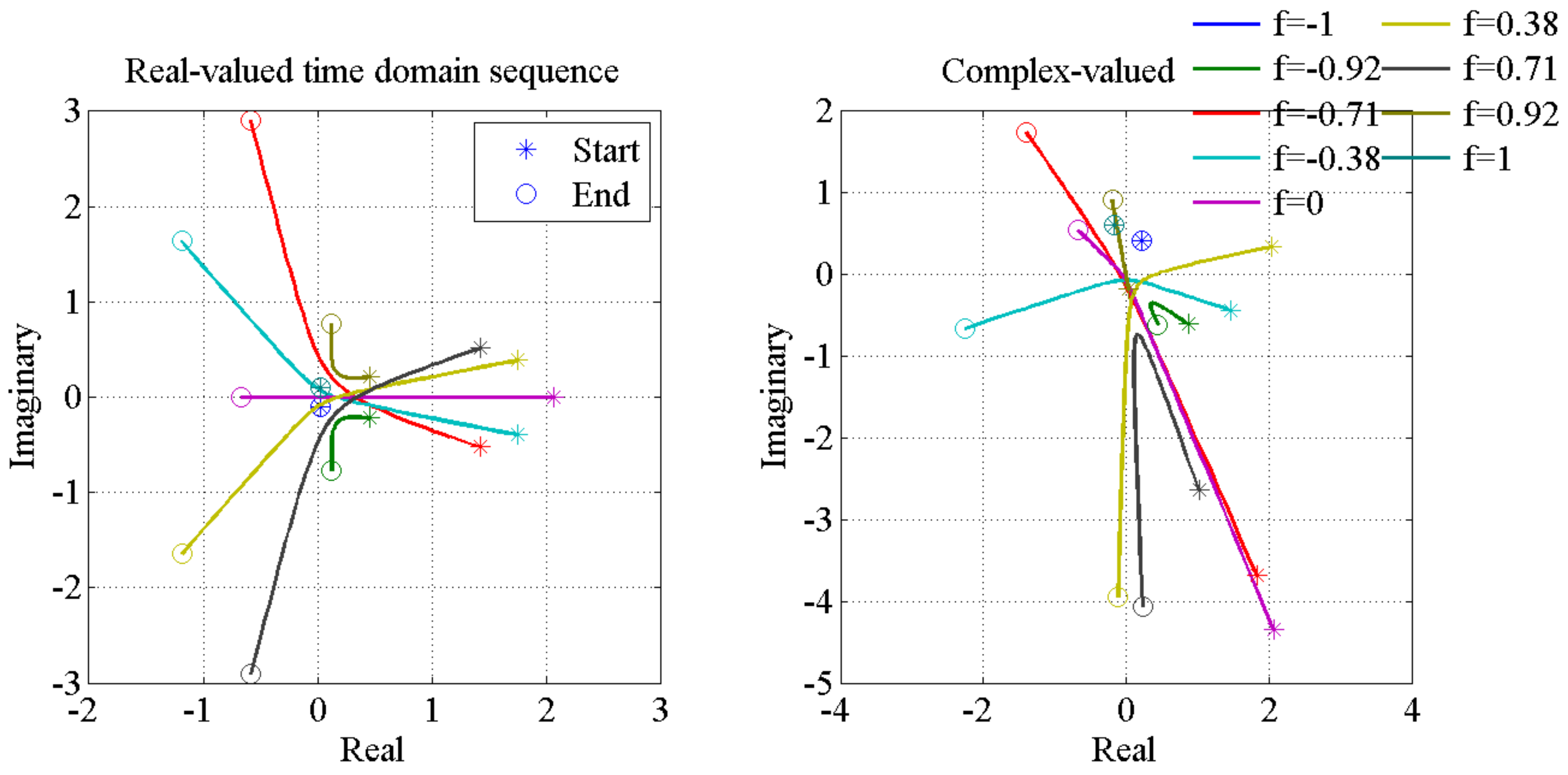
Basis functions Bandwidth Compression: 16 independent amplitude values within 1 μs.



Hermitian Symmetry

Frequency domain trajectories for



Conclusions

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Motivation:

- Applications for ISA: to build a modulation scheme based on ISA for higher spectral efficiency.
    - Patented technology*: Spiral Polynomial Division Multiplexing. (under analysis)

Acknowledgments: work done in partnership with

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Exponential Communication

* **U.S. Patent**: 2013/0251018 A1, **U.S. Patent**: 2012/0263031 A1, **U.S. Patent**: 8995546 B2.