Application of Convex Optimization in Digital Filter Design

• Overview of Digital Filter Design

This world is full of signals[1], which are the necessary ways to realize the communication among people. Nowadays, radio, music, and cell phone signals have become the indispensable part in people's everyday life. Signal processing technology has been developing rapidly since the beginning of this information world.

Signals can be either time-varying variables, which are called continuous/analog signals, or time-varying sequences, which are called discrete/digital signals. From the eyes of engineers, signals usually contain different information of different frequencies, so that we can have both time-domain analysis and frequency-domain analysis of the signals. An analog signal is usually denoted by u(t), where t is the continuous time. An analog signal can have continuous frequency spectrum after we use Fourier Transform (FT) to obtain its frequency domain information. A discrete signal is usually denoted by u(k), where k is an integer number. We can get the frequency-domain analysis of a digital signal by using Z-transform.

Generally speaking, any signal processing function can be called a filter[1], while from the electronics' view, a filter is the device that can discard the parts with frequencies we do not want while keeping useful frequencies of the input signals. Nowadays, filters are widely used in people's real life. We use filter primarily for two reasons. The first reason is that we can use filter to separate information in different frequencies into different channels, so that each channel can transmit a specific frequency range. The second one is that, we can use filter to discard/depress unwanted frequency parts. For example, noise is usually high-frequency signals regard to useful information; therefore, we can use a low-pass filter to depress this noise.

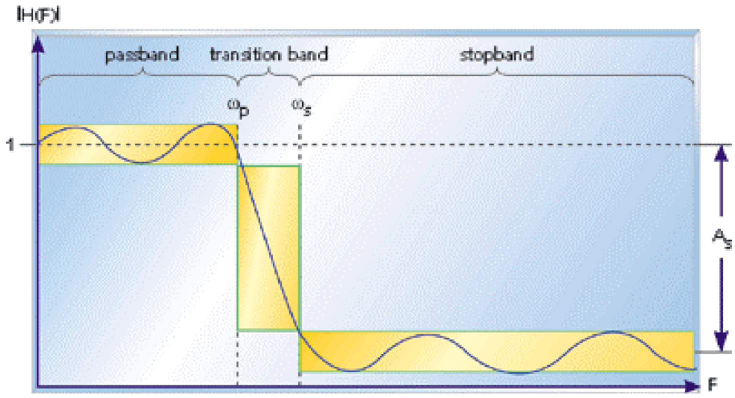


Figure 1, a typical low-pass filter

Figure 1 shows a typical low-pass filter, with its passband from frequency 0 to w_p, and stopband from w_s to infinity. The magnitude of the transfer function H(F) should be as close to 1 as possible in its passband and as close to 0 as possible in its stopband[1]. The solid line of response is not a perfect filter. For perfect filters, the |H(F)|

response should equal to 1 in the passband and equal to 0 in the stopband, however, as we know from DSP courses, this is not possible, so that we can only try our best to approximate the response of the perfect filter.

There are primarily two types of filters[1], analog filters and digital filters. An analog filter is the analog circuit that can filter analog signals, while a digital filter is the digital circuit that can filter digital signals. In this modern world, people are using digital signals and digital filters more and more than analog signals and analog filters. This is because digital systems have two advantages over analog systems. One is that nowadays, especially after the birth of computer technologies, we can use computers to help us process signals, and computers are digital systems. The second reason is that usually digital signals can be much more easily recovered from being noise-corrupted than analog signals. Both in analog filters and in digital filters, we can have a frequency-domain response from frequency 0 to infinity, but because of periodicity and symmetry of the Z-transform, we are only interested in the frequency range [0, π]. For a digital system, we define its transfer function as below:

$$H(z) = \frac{Y(z)}{X(z)} \tag{1}$$

where X(z) is the Z-transform of the input signal, and Y(z) is the Z-transform of the output signal. H(z) is called ***transfer function*** of the system. Although from this equation, we can see that H(z) is the ratio of Y(z) and X(z), however, H(z) is actually a system function, and it does not change with the change of X(z) or Y(z). It is only dependent on the system.

$$y(t) = \sum_{k=0}^{n} h_k u(t-k), \quad t \in \mathbf{Z} \tag{2}$$

$$H(z) = h_0 + h_1 z^{-1} + \cdots + h_n z^{-n}. \tag{3}$$

$$y(t) = \sum_{k=0}^{n} a_k u(t-k) - \sum_{k=1}^{m} b_k y(t-k) \tag{4}$$

$$H(z) = \frac{a_0 + a_1 z^{-1} + \cdots + a_n z^{-n}}{b_0 + b_1 z^{-1} + \cdots + b_m z^{-m}}. \tag{5}$$

Digital filters can also be separated into two types, FIR (finite impulse response) and IIR (infinite impulse response) filters. Equations (2) (3) (4) (5) are FIR filter equation, FIR filter Z-transform, IIR filter equation, and IIR filter Z-transform, respectively. In these equations, y(t) is the output signal at time t, while u(t) is the input signal at time t, and u(t-k) and y(t-k) denote the input and output signal k steps before the time t. Coefficients h's, a's and b's are parameters that are to be decided during the design process. So as we can see, for FIR filters, the present output signal only depends on the input signals, both previous and present, while for IIR filters, the present output not only depends on the input signals, but it also depends on the previous outputs.

• Describe Digital Filter Design as a Convex Optimization Problem

No matter whether it is FIR or IIR digital filter design problems[2][3] we are trying to solve, we can describe the problem as an optimization problem like (6). In (6), H is the transfer function of the system, like H in (3) or (5) for FIR and IIR, respectively. Here, we do not use H(z) any more, but H(w), where $z=e^{jw}$, w is the angel frequency and w=2 π f, f is frequency. After this replacement, $H(e^{jw})$ or H(w) becomes exactly the frequency-domain response of the system, and usually H(w) is complex. $|H(w)|^2$ is called the power spectrum of the system when the input signal is an impulse, while P is the given power spectrum which we are trying to approximate. The actual task of filter design is trying to design a set of coefficients a's b's for IIR filters or h's for FIR filters so that the derived $|H(w)|^2$ of the filter can approximate the given power spectrum P as closely as possible. Because the frequency range we are interested in is continuous from 0 to π, we usually discretize this frequency section by N+1 points and try to let $|H(w)|^2$ of the filter approximate to P at each of these N+1 points. However, this actually can not guarantee that the final result is not only good in these discrete points, but also good between the sections of two adjacent points, so checking the result in the continuous frequency section is necessary after the design. We can obtain the distinction between the given power spectrum and the power spectrum which we derive from $|H(w)|^2$ of the filter at any of these N+1 points. And we want to minimize the maximum of these distinctions. The constraint here $b^Tb=1$ is just normalization. The optimization variables in this problems are a's and b's for IIR design, or h's for FIR design

$$\begin{array}{ll} \text{minimize} & \max_{i=0,\ldots,N} \left| \, |H(e^{j\omega_i})|^2 - P_i \right| \\ \text{subject to} & b^T b = 1, \end{array} \tag{6}$$

What is unlucky, (6) is not convex in a's b's or h's[3], and we have to use the auto-correlation technique to derive a convex problem.

$$u_k = \sum_{i=0}^{n-k} a_i a_{i+k}, \quad v_k = \sum_{i=0}^{m-k} b_i b_{i+k}, \tag{7}$$

$$|H(e^{j\omega})|^2 = \frac{U(\omega)}{V(\omega)} \tag{8}$$

$$\begin{aligned} U(\omega) &= u_0 + 2u_1 \cos\omega + \cdots + 2u_n \cos n\omega, \\ V(\omega) &= v_0 + 2v_1 \cos\omega + \cdots + 2v_m \cos m\omega. \end{aligned} \tag{9}$$

For the general case of (5), also the IIR case, we can bring two new sets of parameters u's and v's, where u_k and v_k here denote k-step auto-correlation of coefficients a's and b's, respectively. From reference paper [2], we know that they are one-to-one corresponding, that is to say, if we know a's and b's, u's and v's are immediately decided, and vice verse. After these parameter replacements, the calculated power spectrum $|H(w)|^2$ becomes the ratio of U and V, where U and V are expressed in (8) and (9), and they are both functions of frequency w. Under these circumstances, the problem becomes a convex optimization problem, described in (10).

$$\begin{array}{ll} \text{minimize} & \max_{i=0,\ldots,N} \left| \dfrac{U(\omega_i)}{V(\omega_i)} - P_i \right| \\ \text{subject to} & v_0 = 1 \\ & u \succeq 0, \quad v \succeq 0, \end{array} \tag{10}$$

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{subject to} & -\gamma V(\omega_i) \le U(\omega_i) - P_i V(\omega_i) \le \gamma V(\omega_i), \quad i = 0, \ldots, N \\ & v_0 = 1 \\ & u \succeq 0, \quad v \succeq 0, \end{array} \tag{11}$$

(11) is an equivalent way to describe the convex optimization problem as (10), but with much more comfort to implement.

Figure (2) shows the result of an experiment[2]. In this experiment, we want to approximate the given power spectrum P (dashed line) by using n=14, m=15 orders of polynomials on numerator and denominator of equation (5), in which we are trying to use an IIR filter design. This experiment shows a very good result. As we can see from Figure (2), although there are some parts of the curve where the solid line and the dashed line do not match very much with each other especially when the curve changes very fast, we still get a good approximation at most of the parts of the curve. This experiment proves that our convex optimization method has the potential to become a method for digital filter design.

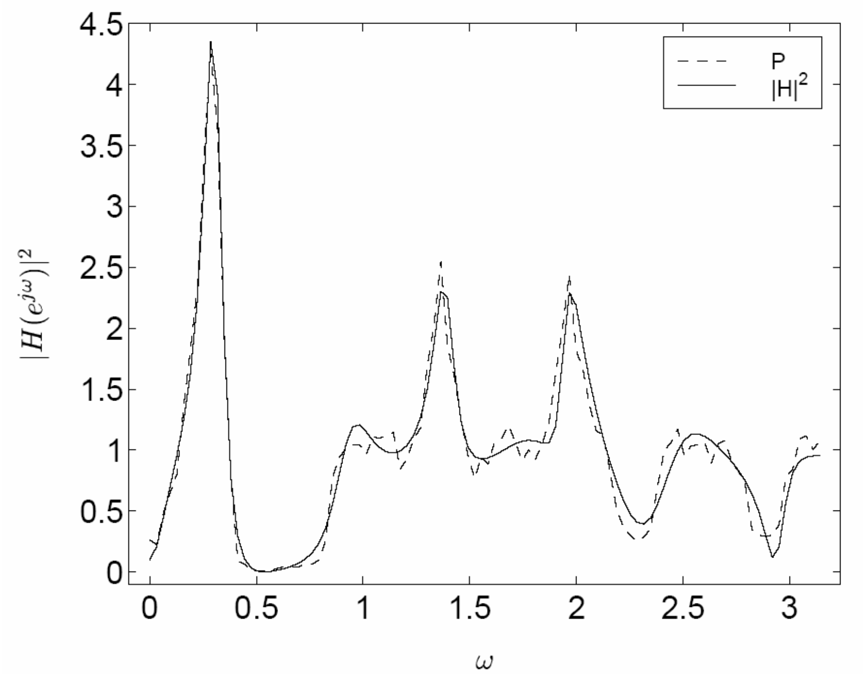


Figure 2, an experiment

For a special case[2], FIR digital filter design, we can also use the above formulas (6) – (11), in which case, b_0=1 and b_1=0, b_2=0, …, b_m=0. However, we can derive it in a simple way. For FIR design problem (2)(3)(6), we can use (12), only one auto-correlation relation to replace the coefficients. The FIR digital filter design problem finally becomes like (13) or (14), and in (14), x's become the optimization variables, and x's are also one-to-one corresponding to h's.

$$x_k = \sum_{i=0}^{n-k} h_i h_{k+i}, \quad k = 0, \ldots, n, \tag{12}$$

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{N} w_k \int_{\alpha_k}^{\beta_k} \left|H(e^{j\omega})\right|^2 d\omega \\ \text{subject to} & L_k \le |H(e^{j\omega})| \le U_k, \quad \omega \in [\alpha_k, \beta_k],\ k = 0, \ldots, N, \end{array} \tag{13}$$

$$\begin{array}{ll} \text{minimize} & \sum_{k=0}^{N} w_k \int_{\alpha_k}^{\beta_k} X(\omega) d\omega \\ \text{subject to} & L_k^2 \le X(\omega) \le U_k^2, \quad \omega \in [\alpha_k, \beta_k],\ k = 0, \ldots, N \\ & X(\omega) \ge 0,\ \omega \in [0, \pi], \\ & X(\omega) = x_0 + 2x_1 \cos\omega + \cdots + 2x_n \cos n\omega. \end{array} \tag{14}$$

in (13) and (14), L_k and U_k denote the lower limit and upper limit of |H(w)| in the kth frequency section. Usually these lower limits and upper limits are decided by power spectrum, and they are parameters to describe the feature of a specific filter. w_k=0 for passbands, and w_k=1 for stopbands, so actually we are trying to limit the magnitude of the transfer function or its square between a pair of given upper and lower limits at each points while trying to minimize the power consumed in stopbands.

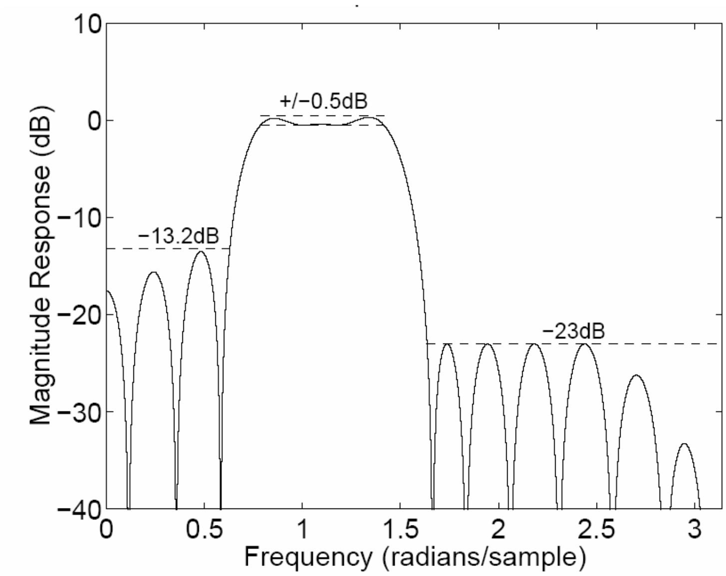


Figure 3. a bandpass filter

• Examples and Comparisons

Figure 3 shows the result of a bandpass filter using convex optimization method[2]. The requirement of this filter is:

Order n=24
stopband [0, 0.2π], Upperbound -13.2db
Passband [0.25π, 0.45π], Gain +-0.5db
Stopband [0.52 π, π], Upperbound -23db

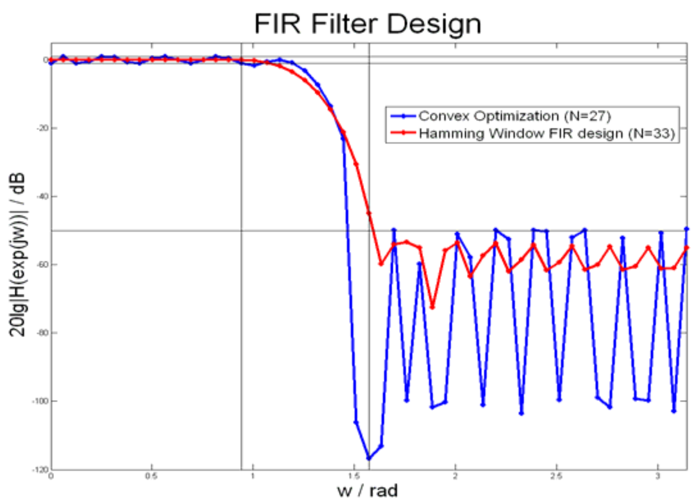
The convex optimization description of this bandpass problem is show in (15)

$$\begin{array}{ll} \text{minimize} & (1/(0.2\pi)) \int_0^{0.2\pi} |H(e^{j\omega})|^2 d\omega + (1/(0.48\pi)) \int_{0.52\pi}^{\pi} |H(e^{j\omega})|^2 d\omega \\ \text{subject to} & 20\log_{10} |H(e^{j\omega})| \le -13.2, \quad \omega \in [0, 0.2\pi] \\ & -0.5 \le 20\log_{10} |H(e^{j\omega})| \le 0.5, \quad \omega \in [0.25\pi, 0.45\pi] \\ & 20\log_{10} |H(e^{j\omega})| \le -23, \quad \omega \in [0.52\pi, \pi]. \end{array} \quad (15)$$

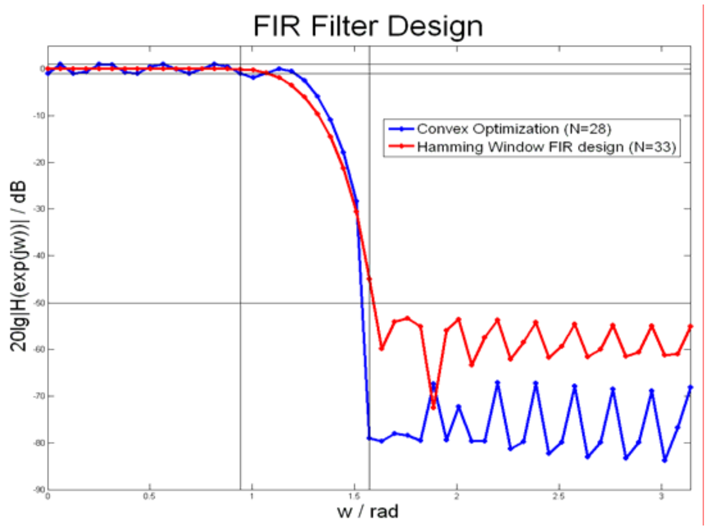
As we can see, the result of this solution is perfect. Both the passband and the two stopbands can satisfy the requirements of this problem.

Figure 4 shows an example of FIR filter design. The indices of this filter are as below:

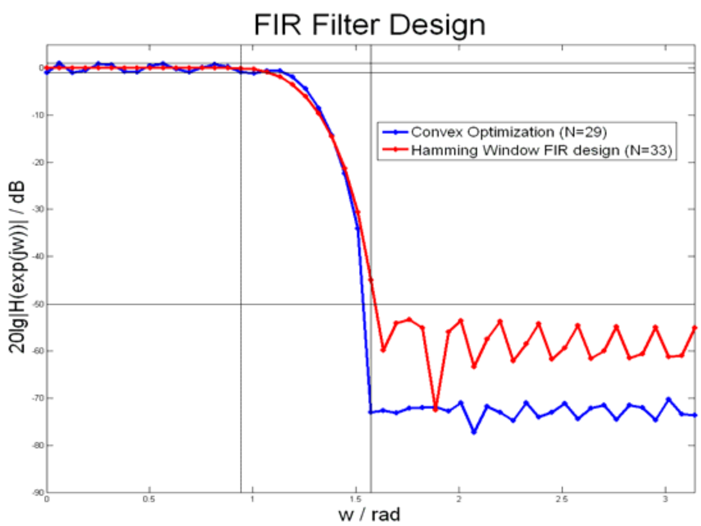
passband [0,0.3π], + - 2 db; stopband [0.5π, π], < -50 db



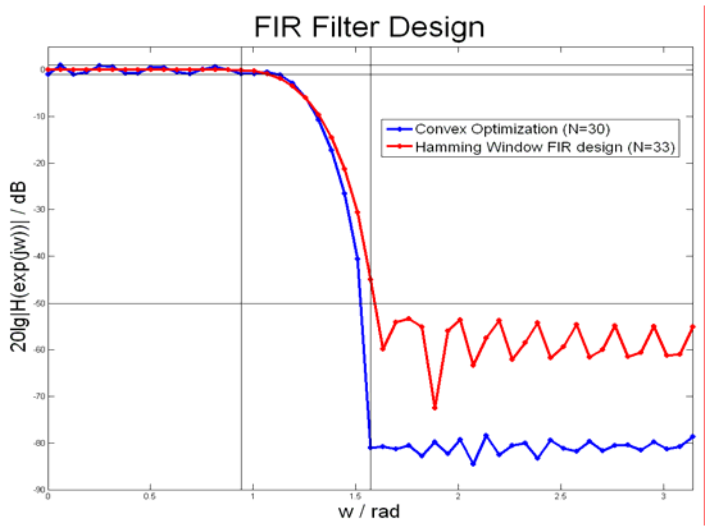
N=27



N=28



N=29



N=30

Figure 4, a FIR filter design using convex optimization method

In Figure 4, we display the results from using conventional FIR design method (red line) and the results from using convex optimization method when the polynomial order is 27, 28, 29 and 30 respectively. When using the conventional method, we have to use a polynomial of order 33 to approximate the requirements. However, when using the convex optimization method, we just use an order of at most 30 to get the suitable results. And these results all have a much better improvement especially in the stopband, and with the increase of the order, the result is getting better and better.

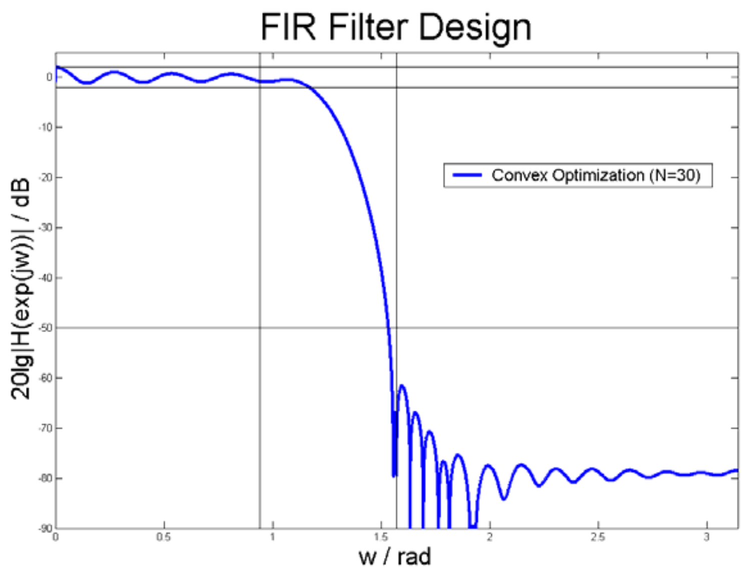


Figure 5, continuous response of the FIR filter in Figure 4

As we stated, we have to check the final result after the completion of our design, because we have to make sure that not only in the discrete points we can get the good result, but in those sections between points, we can also get a good result. Figure 5 shows the result along the continuous frequency. The most important difference between curve in Figure 5 and curve N=30 in Figure 4 is that there exists much more vibration in section [1.5, 2] in Figure 5 than in Figure 4. However, this does not affect the satisfaction of requirements in this problem, therefore, N=30 is a very good design when using convex optimization method.

Figure 6 gives an example of IIR filter design when using convex optimization method. The indices of this low-pass filter are shown as below:

passband $[0, 0.318\pi]$, + - 3 db; stopband $[0.8\pi, \pi]$, < -15 db

The left part of Figure 6 shows the comparison between the results derived from two conventional methods (Bi-Linear transform method, and Impulse invariant method) and the results obtained from convex optimization method. We can see that, when using an order of n=3 & m=3, we actually can get much better results than using conventional methods. In the passband [0, 1], we almost get a flat response near |H(w)|=1, while the results from using the other two conventional methods both show some vibration away from |H(w)|=1. In addition, in the stopband, we get a much better depression for the convex optimization method than the two conventional methods. Again, we have to make a continuous check about the result, and the result is shown in the right part. It is really a reliable IIR filter.

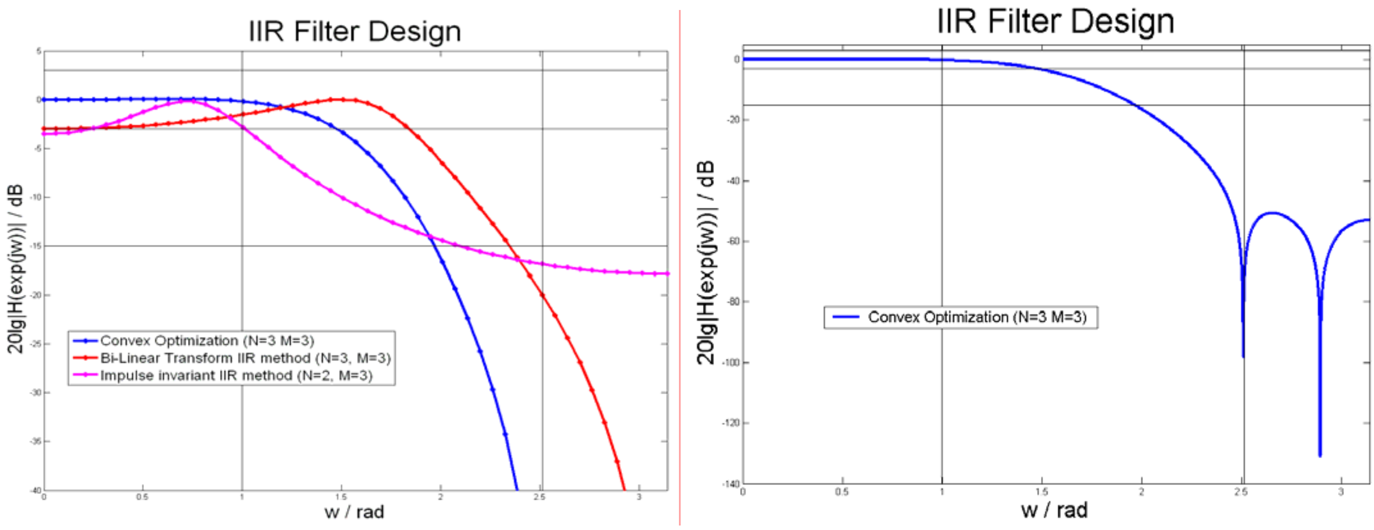


Figure 6, a IIR filter design using convex optimization method

• Conclusion

We can see from the above examples and comparisons that, convex optimization method can be used in digital filter design problems, and we can get the results similar to or better than conventional methods. Although FIR & IIR filter design problems are not convex themselves, we can convert them into a convex one after using an auto-correlation transformation. For some specific problems, we can achieve the same or similar goals when using less order of polynomials, either in IIR or FIR.

• Reference

[1] ***Signals & Systems (second book),*** P171 – P251, J. Zheng, 2000, High Education Press

[2] ***Convex Optimization Problems Involving Finite Autocorrelation Sequences***, B. Alkire and L. Vandenberghe, In *Mathematical Programming, Series A*, Volume 93, Issue 3, pages 331-359, 2002.

[3] ***Applications of Convex Optimization in Signal Processing and Digital Communication***, Z. Luo, In Mathematical programming, 2003, vol. 97, pp. 177-207.