PAPER

Link Capacity Assignment in Packet-Switched Networks: The Case of Piecewise Linear Concave Cost Function

Suwan RUNGGERATIGUL†, *Member and* Sawasd TANTARATANA†, *Nonmember*

SUMMARY In this paper, we study the link capacity assignment problem in packet-switched networks (CA problem) focusing on the case where link cost function is a piecewise linear concave function. This type of cost function arises in many communication network design problems such as those arising from developments in communication transmission technologies. It is already known that the method of link set assignment is applicable for solving the CA problem with piecewise linear convex cost function. That is, each link in the network is assigned to one of a group of specific sets, and checked for link set contradiction. By extending the method of link set assignment to the case of piecewise linear concave cost function, an important characteristic of the optimal solution of the CA problem is derived. Based on this characteristic, the non-differentiable link cost function can be treated as a differentiable function, and a heuristic algorithm derived from the Lagrange multiplier method is then proposed. Although it is difficult to determine the global optimum of the CA problem due to its non-convexity, it is shown by numerical results that the solution obtained from the proposed algorithm is very close to the global optimum. Moreover, the computation time is linearly dependent on the number of links in the problem. These performances show that the proposed algorithm is very efficient in solving the CA problem, even in the case of large-scale networks.

key words: *communication networks and services, packet-switched networks, network design, link capacity assignment, non-linear programming, algorithm*

1. Introduction

Link capacity assignment problem in packet-switched networks (CA problem) has been studied widely since the early days of ARPANET's appearance [1], [2]. This problem is concerned with means to determine capacity of links that minimize network cost, subject to some constraints, such as the upper limit of average packet delay. The CA problem can be solved when network information, including network topology, routing pattern, etc., is given. The problem has been studied for many kinds of network model, e.g., network with single class or multi-class packets, network with several kinds of design parameter: packet delay or packet loss rate or both of them, network with continuous or discrete link capacity model, etc.

Many types of link cost function have been considered in the CA problem. For example, some fundamental mathematical functions including linear, exponential, logarithmic, are examined to see the behavior of link capacity assignment due to the effect of these cost functions [2]. A piecewise linear convex cost function has also been considered as a link cost model in the case that the CA problem is solved with the consideration of existing network, where the per-unit cost of using existing capacity is less than that of installing new capacity [3].

In general cases of communication network design, it is proper to consider link cost function as a concave function [2], [4]. This is because the economy of scale is often present in communication resources. For example, we can obtain a cost function from the envelope of several cost functions due to the development in communication transmission technologies [5], [6], the development in switching technologies [7], etc.

In this paper, we study the CA problem in packet-switched networks where link cost function is piecewise linear concave. This is the case when each link in the network is implemented by selecting a link type from several alternatives whose cost function is linear. Clearly, this type of link cost function is non-differentiable. To solve problems with this type of cost function, the method of approximating the non-differentiable cost function by a differentiable function has been proposed [8]. Although this method yields a solution within approximation error, there is no guarantee that the solution obtained from this method is optimum when link cost function is not convex. Moreover, lengthy time is needed to determine the approximation function.

The main objective of this paper is to clarify the characteristic of optimal solution of the CA problem. By applying the method of link set assignment [3], we can derive an important characteristic of the optimal solution. Based on this characteristic, there is no need to perform approximation on the non-differentiable link cost function as in the above method, and the cost function can be treated as a differentiable function. As a result, conventional methods such as Lagrange multiplier method can be applied to the problem.

It should be noted that the CA problem taking into consideration an existing network in the case of long-term design (the per-unit cost of installing new capacity is less than that of using existing capacity) is a special case of the CA problem studied in this paper.

The rest of this paper is organized as follows. First,

Manuscript received January 8, 1999.

Manuscript revised March 31, 1999.

†The authors are with Sirindhorn International Institute of Technology, Thammasat University, Thailand.

a packet-switched network model used in this paper and the detail of link cost function are discussed in Sect. 2. The CA problem is then formulated in Sect. 3, where the concept of link set assignment for solving the CA problem is given. In Sect. 4, an important characteristic of the optimal solution of the CA problem is derived. Based on this characteristic, a heuristic algorithm derived from the Lagrange multiplier method is presented in Sect. 5. Some numerical results are given in Sect. 6, and conclusions are given in Sect. 7.

2. Network Model

In this section, the model of packet-switched networks considered in this paper is discussed. Then a piecewise linear concave link cost function is introduced.

2.1 Model Description

The packet-switched network model used in this paper is the same as the one used in the early work of packet-switched network design [1]–[3]. Packet arrival process on each link is assumed to be Poisson, and packet length is assumed to be negative exponentially distributed. From this assumption, we can model each link as an $M/M/1$ system with infinite buffer, and the average packet delay throughout the network T can be given as in Eq. (1).

$$T = \frac{1}{\gamma} \sum_{L} \frac{f_i}{C_i - f_i}, \tag{1}$$

where L is the set of links in the network, f_i is the traffic flow on link i (in bits/second), C_i is the capacity of link i (in bits/second), and γ is the overall traffic in the network (in packets/second). The value of link capacity C_i is assumed to be continuous, and can be set as an arbitrary positive value. For simplicity, node cost (e.g., cost of switching facility, etc.) is not taken into account in this model. Only link cost is considered, and the total network cost D is defined as the sum of all link costs, i.e.,

$$D = \sum_{L} D_i, \tag{2}$$

where D_i is the cost of link i.

2.2 Link Cost Function

We assume that there are p alternatives of link types that can be selected to implement each link in the network, where $p > 1$ and the cost function of each link type is linear. Consequently, the cost function of link type k for link i can be given as

$$D_{ki} = d_{ki} C_i + r_{ki}, \qquad 0 \le k \le p-1, \tag{3}$$

where d_{ki} and r_{ki} are respectively the per-unit cost and

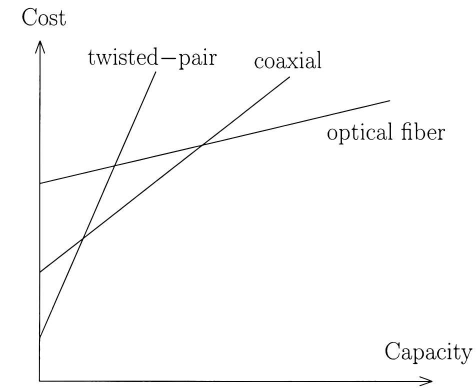


Fig. 1 An example of practical link type selection.

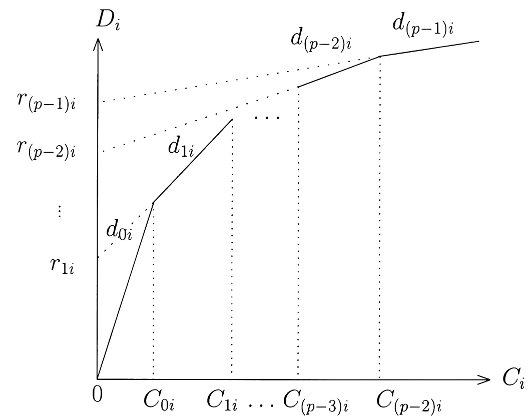


Fig. 2 A piecewise linear concave link cost function.

the start-up cost of link type k for link i.

Hence, the link cost function is the envelope of cost functions of all link types, which is a piecewise linear function of link capacity as

$$D_i = \min_k (d_{ki} C_i + r_{ki}), \qquad 0 \le k \le p-1. \tag{4}$$

Since we are going to investigate the CA problem with piecewise linear concave cost function, we consider the following case:

$$d_{ui} > d_{vi}, \qquad r_{ui} < r_{vi}, \tag{5}$$

when $0 \le u, v \le p-1$ and $u < v$. Without loss of generality, we can assume that $r_{0i} = 0, \forall i \in L$.

A practical example with three alternatives of link types is given in Fig. 1.

As a whole, the expression of the link cost function can be given as in Eq. (6), which is depicted in Fig. 2.

$$D_i = \begin{cases} d_{0i} C_i &, \quad C_i \le C_{0i}, \\ d_{1i} C_i + r_{1i} &, \quad C_{0i} \le C_i \le C_{1i}, \\ \vdots & \\ d_{(p-1)i} C_i + r_{(p-1)i} &, \quad C_i \ge C_{(p-2)i}. \end{cases} \tag{6}$$

3. Link Capacity Assignment Problem

In this section, the CA problem for packet-switched networks is formulated as a non-linear programming problem, and then the concept of link set assignment for solving the problem is discussed.

3.1 Problem Formulation

We formulate the CA problem as follows:

CA Problem

Given: $\{f_i\}$

Minimize: $D = \sum_{L} D_i$

Design variables: $\{C_i\}$

Subject to: $T = \dfrac{1}{\gamma} \sum_{L} \dfrac{f_i}{C_i - f_i} \leq T_{\max}$,

$C_i - f_i > 0, \quad \forall i \in L$

where $T_{\max}$ is the constraint value of T allowed in the network.

In the above CA problem, $\{f_i\}$ is the traffic pattern in the network, and $f_i > 0, \forall i \in L$. Link capacity $\{C_i\}$ is determined so that the total network cost D is minimized. Design constraints are the upper limit of packet delay, and the relationship between link flow and link capacity. Obviously, link capacity must be greater than link flow so that packet delay will not grow indefinitely.

3.2 The Method of Link Set Assignment

The CA problem formulated above is a non-linear programming problem. Many conventional methods such as Lagrange multiplier method are known as efficient methods for solving non-linear programming problems, e.g., the CA problem studied in the earlier work [2].

However, since the link cost function D_i given in Eq. (6) is not differentiable with respect to C_i, Lagrange multiplier method cannot be applied to the CA problem. As a method to alleviate this type of network design problem, the non-differentiable function is approximated by a differentiable function as shown in Fig. 3 [8]. By using an approximated differentiable link cost function, Lagrange multiplier method can be applied to the CA problem. Although, this provides a solution to the problem within approximation error, there is no guarantee that the solution obtained is an optimal solution when the link cost is not convex. Moreover, lengthy time is needed for calculating approximation function of all cost functions.

When the cost function is piecewise linear convex, a method of link set assignment has been proposed to solve the CA problem to obtain an optimal solution [3]. In this paper, we extend this method so that it is applicable to the CA problem with piecewise linear concave cost function. The concept of the method is as follows.

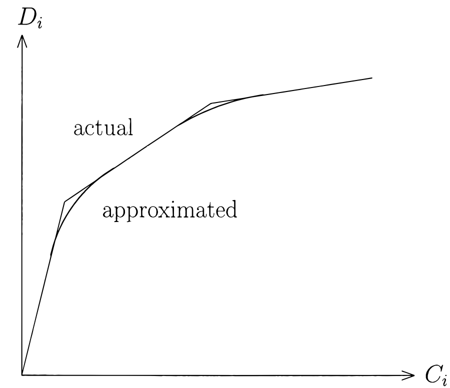


Fig. 3 Approximation of non-differentiable function.

D_i given in Eq. (6) can be considered as a differentiable function, if we assign each link i to be a member of one of the following $2p-1$ sets:

$$\begin{aligned}
L_0 &= \{\, i \mid C_i < C_{0i} \},\\
L_1 &= \{\, i \mid C_i = C_{0i} \},\\
L_2 &= \{\, i \mid C_{0i} < C_i < C_{1i} \},\\
L_3 &= \{\, i \mid C_i = C_{1i} \},\\
&\vdots\\
L_{2p-4} &= \{\, i \mid C_{(p-3)i} < C_i < C_{(p-2)i} \},\\
L_{2p-3} &= \{\, i \mid C_i = C_{(p-2)i} \},\\
L_{2p-2} &= \{\, i \mid C_i > C_{(p-2)i} \}.
\end{aligned}$$

This is because

$$\frac{dD_i}{dC_i} = \begin{cases} d_{0i} &, \quad i \in L_0, \\ d_{1i} &, \quad i \in L_2, \\ \quad\vdots & \\ d_{(p-2)i} &, \quad i \in L_{2p-4}, \\ d_{(p-1)i} &, \quad i \in L_{2p-2}. \end{cases} \tag{7}$$

Note that link i that belongs to L_{2k+1} $(0 \leq k \leq p-2)$ is excluded from the CA problem since its capacity C_i is set to be a constant value of C_{ki}.

With the above link set assignment, Lagrange multiplier method can be applied to the CA problem, and the solution of $\{C_i\}$ can be given as in Eq. (8) [3].

$$C_i = f_i + \frac{\sum_{L_E} \sqrt{f_j d_j}}{\left(\gamma\, T_{\max} - \sum_{L_O} \frac{f_j}{X_j - f_j}\right)} \sqrt{\frac{f_i}{d_i}}, \quad \forall i \in L_E, \tag{8}$$

where

$$\begin{aligned} L_E &= L_0 \cup L_2 \cup \cdots \cup L_{2p-4} \cup L_{2p-2}, \\ L_O &= L_1 \cup L_3 \cup \cdots \cup L_{2p-5} \cup L_{2p-3}, \\ X_j &= C_{kj}, \text{ when } j \in L_{2k+1}, \quad 0 \le k \le p-2, \end{aligned}$$

and d_i is equal to $\frac{dD_i}{dC_i}$ given in Eq. (7).

The value of C_i calculated from Eq. (8) is optimum for the CA problem under the relevant link set assigned at the beginning. However, the value has to be checked whether it contradicts its assigned link set or not, e.g., for $i \in L_0$, C_i that is greater than C_{0i} is thought to be a contradiction. If any contradiction occurs, the solution obtained will not be feasible and other combinations of link set have to be examined until a feasible solution with minimum network cost is found.

Although the above method gives global optimal solution of the CA problem, it is very time-consuming since we have to examine all link set combinations. Thus, a more efficient algorithm is required.

4. Characteristic of the Optimal Solution of the CA Problem

To find a way to solve the CA problem, we first review the modified method of link set assignment given in Ref. [3]. For simplicity, we first discuss the case $p = 2$, and then extend the discussion to the case $p > 2$.

4.1 The Case $p = 2$

For this case, we have

$$\begin{aligned} L_0 &= \{\, i \mid C_i < C_{0i}\}, \\ L_1 &= \{\, i \mid C_i = C_{0i}\}, \\ L_2 &= \{\, i \mid C_i > C_{0i}\}, \end{aligned}$$

and

$$C_i = f_i + \frac{\sum_{L_0 \cup L_2} \sqrt{f_j d_j}}{\left(\gamma\, T_{\max} - \sum_{L_1} \frac{f_j}{C_{0j} - f_j}\right)} \sqrt{\frac{f_i}{d_i}}, \quad \forall i \in L_0 \cup L_2, \tag{9}$$

where $d_i = d_{0i}$ if $i \in L_0$ and $d_i = d_{1i}$ if $i \in L_2$.

Since each link in the network certainly belongs to L_0, L_1, or L_2, we can select one of the three sets for each link and assign the link to that set. Then checking the contradiction between the assigned link set and the calculated link capacity will give us a way to decide that the link can belong to the set that it was assigned or not.

In the case of convex cost function ($d_{0i} < d_{1i}$, $\forall i \in L$), each link is first assigned to L_0 and if contradiction occurs, it is concluded that the link with contradiction cannot belong to L_0. In the next step, each link is set to L_2 and if contradiction occurs, we can say that the link with contradiction cannot belong to L_2. In the case that a link cannot belong to neither L_0 nor L_2, it must belong to L_1. By repeatedly examining link for L_1 until no more link has to be assigned to it, each link in the network will belong to a proper set, and link capacity can be computed by using Eq. (9). It is proven that the final solution by the method of link set assignment is a global optimal solution for the case of convex cost function [3].

Next, we investigate that how the above method can deal with the CA problem with concave cost function ($d_{0i} > d_{1i}$, $\forall i \in L$). First, we assign each link to L_0 and use Eq. (9) to determine $\{C_i\}$, then check whether there is any contradiction or not, i.e., there is link i that $C_i \ge C_{0i}$ or not. Unlike the case of convex cost function, we focus on the links without contradiction. This is because we can show that this kind of links can belong to L_0 without any contradiction although link set combination is changed. See Appendix for the proof. Again when each link is set to L_2, we can say that the links with no contradiction can belong to L_2.

By the above concept, we have the following procedure for assigning a proper set to each link. Let's introduce two parameters for link i: e_{0i} and e_{2i}. The values of the two parameters are set as follows: e_{0i} will be 0 if link i is assigned to L_0 and no contradiction occurs, and be 1 if contradiction exists. e_{2i} will be 0 if link i is assigned to L_2 and no contradiction occurs, and be 1 if contradiction exists. Then we consider the following four cases:

1. $e_{0i} = 0$ and $e_{2i} = 1$,
2. $e_{0i} = 1$ and $e_{2i} = 0$,
3. $e_{0i} = 0$ and $e_{2i} = 0$,
4. $e_{0i} = 1$ and $e_{2i} = 1$.

In case 1, we can let link i belong to L_0 since there is no contradiction as mentioned above. Although in global optimal solution of the problem, link i may not belong to L_0, we deal with the link in a greedy manner to obtain a feasible solution since it is guaranteed that there will be no contradiction even the sets of other links are changed. Moreover, due to the non-convexity of the CA problem, its global optimum cannot be determined easily, so we have to try to find its local optimum instead. As same as case 1, link i that falls into case 2 is then assigned to L_2.

In case 3, since link i can belong to either L_0 or L_2

without any contradiction, we can apply any method to select the set for this type of links, e.g., L_0 and L_2 are randomly chosen.

Finally, for link of case 4, we do not assign this type of links to any set, but repeatedly do the above procedure again until all links have their proper set.

There will be no problem if links of case 4 do not exist at the final stage of the examining, and each link in the network will then be assigned to L_0 or L_2. However, if this kind of links exist, the question that they can be assigned to L_1 or not will arise. To answer this question we introduce the following theorem.

Theorem 1: In the optimal solution of the CA problem with $p = 2$,

$$L_1 = \emptyset.$$

Proof: We define the following sets and notations.

$$L'_0 = \{i \mid i \text{ is the link already assigned to } L_0\},$$
$$L'_2 = \{i \mid i \text{ is the link already assigned to } L_2\},$$

$$b_0 = \sum_{L'_0} \sqrt{f_j d_{0j}}, \quad b'_0 = \sum_{L-(L'_0 \cup L'_2)} \sqrt{f_j d_{0j}},$$
$$b_1 = \sum_{L'_2} \sqrt{f_j d_{1j}}, \quad b'_1 = \sum_{L-(L'_0 \cup L'_2)} \sqrt{f_j d_{1j}}.$$

Next, assume that there exists link i such that $e_{0i} = 1$ and $e_{2i} = 1$. Then we have

$$C_i = f_i + \frac{b_0 + b_1 + b'_0}{\gamma\, T_{\max}} \sqrt{\frac{f_i}{d_{0i}}} \geq C_{0i}, \tag{10}$$

$$C_i = f_i + \frac{b_0 + b_1 + b'_1}{\gamma\, T_{\max}} \sqrt{\frac{f_i}{d_{1i}}} \leq C_{0i}. \tag{11}$$

Manipulating (10) and (11) yields

$$\alpha_i\ (b_0 + b_1 + b'_0) \geq \sqrt{f_i d_{0i}}, \tag{12}$$

$$\alpha_i\ (b_0 + b_1 + b'_1) \leq \sqrt{f_i d_{1i}}, \tag{13}$$

where $\alpha_i = \dfrac{f_i}{\gamma\, T_{\max}\ (C_{0i} - f_i)}$.

At the final stage of the examining, (12) and (13) are valid for $\forall i \in L - (L'_0 \cup L'_2)$, so we can take the summation of (12) and (13) over the set $L - (L'_0 \cup L'_2)$. This gives

$$\sum_{L-(L'_0 \cup L'_2)} \alpha_i\ (b_0 + b_1 + b'_0) \geq b'_0, \tag{14}$$

$$\sum_{L-(L'_0 \cup L'_2)} \alpha_i\ (b_0 + b_1 + b'_1) \leq b'_1. \tag{15}$$

From (14) and (15), we obtain

$$\frac{b_0 + b_1 + b'_0}{b_0 + b_1 + b'_1} \geq \frac{b'_0}{b'_1}. \tag{16}$$

After manipulating (16), we get

$$b'_1 \geq b'_0. \tag{17}$$

However, this contradicts to the fact that

$$d_{0i} > d_{1i}, \qquad \forall i \in L.$$

Thus, there cannot be any link i that $e_{0i} = 1$ and $e_{2i} = 1$ at the final stage of examining, and all links in the network will belong to L_0 or L_2. This implies that

$$L_1 = \emptyset. \tag{18}$$

□

4.2 The Case $p > 2$

Next, the method of link set assignment and the above procedure are extended to the case $p > 2$.

First, we define E_i as the number of sets among $L_0, L_2, \ldots, L_{2p-4}, L_{2p-2}$ that make contradictions when link i is assigned to. Then, $E_i \neq p$ means there is at least one set among $L_0, L_2, \ldots, L_{2p-4}, L_{2p-2}$ that makes no contradiction with link i. Consequently, link i can be assigned to that set without any contradiction. If the number of this kind of sets is more than one, we can take any criteria to select one set from the possible sets, e.g., the set is chosen in random.

In the case that $E_i = p$, we do not assign link i to any set, but repeatedly do the procedure of link set assignment until each link in the network is assigned to a proper set.

Again, the above procedure will work successfully if there is no link i that $E_i = p$ at the final stage of examining, and each link in the network will be assigned to one of the following sets: $L_0, L_2, \ldots, L_{2p-4}$, and L_{2p-2}. Therefore we come to the question: Is there any link i that $E_i = p$ at the final stage of the examining? To answer this question, we first introduce the following lemma.

Lemma 1: If there is link i that $E_i = p$ at the final stage of examining, we cannot have the following cases at the same time.

1. $C_i \geq C_{mi}$ when $i \in L_{2m}$,
2. $C_i \leq C_{mi}$ when $i \in L_{2m+2}$,

where $0 \leq m \leq p - 2$.

Proof: We define the following sets and notations where $0 \leq k \leq p - 1$.

$$L'_{2k} = \{i \mid i \text{ is the link already assigned to } L_{2k}\},$$

$$b_k = \sum_{L'_{2k}} \sqrt{f_j d_{kj}}, \qquad b'_k = \sum_{L'} \sqrt{f_j d_{kj}},$$

$$B = b_0 + b_1 + b_2 + \cdots + b_{p-2} + b_{p-1},$$
$$L' = L - (L'_0 \cup L'_2 \cup L'_4 \cup \cdots \cup L'_{2p-4} \cup L'_{2p-2}).$$

When link i is assigned to set L_{2m} and L_{2m+2}, we

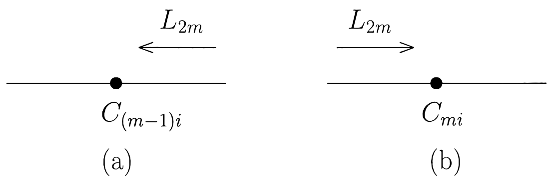


Fig. 4 Representation of link set contradictions.

assume that link set contradictions occur in the case that $C_i \geq C_{mi}$ when $i \in L_{2m}$, and $C_i \leq C_{mi}$ when $i \in L_{2m+2}$. Then we have

$$C_i = f_i + \frac{B + b'_m}{\gamma\, T_{\max}} \sqrt{\frac{f_i}{d_{mi}}} \geq C_{mi}, \tag{19}$$

$$C_i = f_i + \frac{B + b'_{m+1}}{\gamma\, T_{\max}} \sqrt{\frac{f_i}{d_{(m+1)i}}} \leq C_{mi}. \tag{20}$$

From (19) and (20), we get

$$\alpha_i\ (B + b'_m) \geq \sqrt{f_i d_{mi}}, \tag{21}$$

$$\alpha_i\ (B + b'_{m+1}) \leq \sqrt{f_i d_{(m+1)i}}, \tag{22}$$

where $\alpha_i = \dfrac{f_i}{\gamma\, T_{\max}\ (C_{mi} - f_i)}$.

At the final stage of the examining, (21) and (22) are valid for $\forall i \in L'$, so we can take the summation of (21) and (22) over the set L'. This yields

$$\sum_{L'} \alpha_i\ (B + b'_m) \geq b'_m, \tag{23}$$

$$\sum_{L'} \alpha_i\ (B + b'_{m+1}) \leq b'_{m+1}. \tag{24}$$

From (23) and (24), we obtain

$$b'_{m+1} \geq b'_m. \tag{25}$$

However, this contradicts to the fact that

$$d_{mi} > d_{(m+1)i}, \qquad \forall i \in L.$$

Hence, the link set contradiction at L_{2m} and L_{2m+2} cannot be $C_i \geq C_{mi}$ and $C_i \leq C_{mi}$, respectively, at the same time. □

When link i is assigned to $L_{2m},\ 1 \leq m \leq p-3$, and contradiction occurs, we can have two cases:

$$C_i \leq C_{(m-1)i} \text{ or } C_i \geq C_{mi}.$$

We represent the above link set contradictions by arrows pointing toward $C_{(m-1)i}$ and C_{mi}, respectively, as in Fig. 4.

From Lemma 1, we cannot have the contradictions at L_{2m} and L_{2m+2} as two arrows pointing toward $C_i = C_{mi}$ at the same time, i.e., the case that is shown in Fig. 5(a). As a consequence, if there are contradictions at both L_{2m} and L_{2m+2}, the possible cases must be those given in Fig. 5(b), (c), or (d).

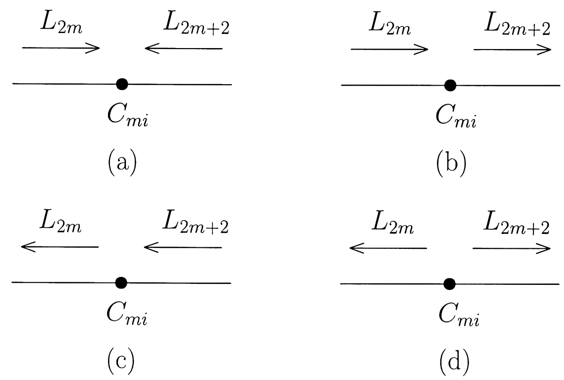


Fig. 5 Link set contradictions at L_{2m} and L_{2m+2}.

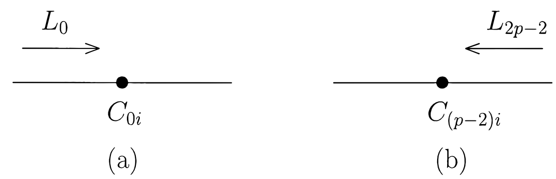


Fig. 6 Link set contradictions at L_0 and L_{2p-2}.

From Lemma 1 and arrow representation of link set contradiction, we introduce the following theorem to give an answer for the question that there is any link i with $E_i = p$ at the final stage of link set examining or not.

Theorem 2: In the optimal solution of the CA problem with $p > 2$,

$$L_{2k+1} = \emptyset, \qquad 0 \leq k \leq p-2.$$

Proof: Assume that at the final stage of link set examining, there are links that have contradictions at all of the sets: $L_0, L_2, L_4, \ldots, L_{2p-4}$, and L_{2p-2}. In other words, there are links i that $E_i = p$.

Let's consider the contradictions at L_0 and L_{2p-2}. Clearly, we must have $C_i \geq C_{0i}$ when $i \in L_0$, and $C_i \leq C_{(p-2)i}$ when $i \in L_{2p-2}$. By arrow representation, these two cases can be depicted as in Fig. 6.

First, we investigate the link set contradiction at L_0. This contradiction can be represented by an arrow pointing to $C_i = C_{0i}$. To make link set contradictions occur at $L_2, L_4, \ldots, L_{2p-6}, L_{2p-4}$, we must have arrows pointing toward the points C_i equal to $C_{1i}, C_{2i}, \ldots, C_{(p-3)i}, C_{(p-2)i}$, respectively. However, the contradiction at L_{2p-2} is an arrow pointing toward $C_i = C_{(p-2)i}$ as shown in Fig. 7(a). By Lemma 1, this is impossible.

Next, we consider the contradiction at L_{2p-2}. This contradiction is represented by an arrow pointing to $C_i = C_{(p-2)i}$. To make the contradictions occur at $L_{2p-4}, L_{2p-6}, \ldots, L_4, L_2$, we must have arrows pointing toward the points C_i equal to

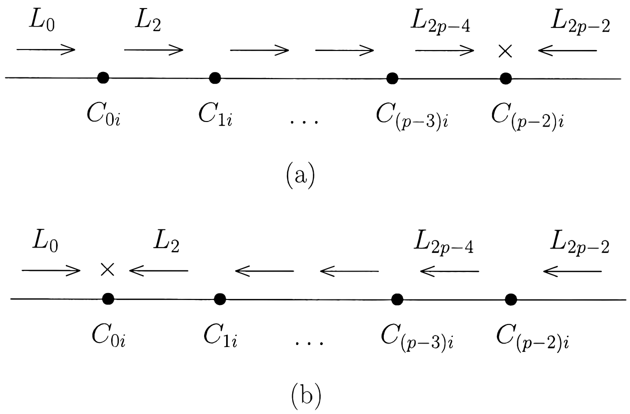


Fig. 7 The impossibility of link set contradictions.

$C_{(p-3)i}, C_{(p-4)i}, \dots, C_{1i}, C_{0i}$, respectively. However, the contradiction at L_0 is an arrow pointing toward $C_i = C_{0i}$ as shown in Fig. 7(b). By Lemma 1, this is also impossible.

Thus, at the final stage, there is no link that has contradictions at $L_0, L_2, L_4, \dots, L_{2p-4}$, and L_{2p-2}, i.e., there is no link i that $E_i = p$. This also means that each link in the network is assigned to one of the following sets: $L_0, L_2, L_4, \dots, L_{2p-4}$, and L_{2p-2}.

Hence,

$$L_1 = L_3 = \dots = L_{2p-5} = L_{2p-3} = \emptyset, \tag{26}$$

or

$$L_{2k+1} = \emptyset, \qquad 0 \le k \le p-2. \tag{27}$$

□

5. Algorithm for the CA Problem

Based on the fact that $L_{2k+1} = \emptyset$, $0 \le k \le p-2$ in the optimal solution of the CA problem, we can exclude the point $C_i = C_{ki}$ from the cost function of all links in the network, and the link cost function can then be treated as a differentiable function, where $\dfrac{dD_i}{dC_i}$ can be expressed as in Eq. (7).

Clearly, we have no need to approximate the non-differentiable function by a differentiable function as that was done in Ref. [8]. As a result, the Lagrange multiplier method can be directly applied to the CA problem as in the following algorithm.

ALGORITHM

Step 1 Initialize d_i as d_{ki}, $0 \le k \le p-1$ by any method, $\forall i \in L$.

Set ϵ as a small positive value for using as algorithm termination parameter.

Determine the initial value of Lagrange multiplier β' by

$$\beta' = \left(\frac{\sum_L \sqrt{f_i d_i/\gamma}}{T_{\max}} \right)^2.$$

Step 2 Determine C_i by

$$C_i = f_i + \sqrt{\frac{\beta' f_i}{\gamma\ d_i}}, \qquad \forall i \in L.$$

Step 3 Set d_i, $\forall i \in L$ as follows:

$$\begin{aligned} d_i &= d_{0i} \text{ if } C_i < C_{0i}, \\ d_i &= d_{1i} \text{ if } C_{0i} < C_i < C_{1i}, \\ &\vdots \\ d_i &= d_{(p-1)i} \text{ if } C_i > C_{(p-2)i}. \end{aligned}$$

Step 4 Determine Lagrange multiplier β by

$$\beta = \left(\frac{\sum_L \sqrt{f_i d_i/\gamma}}{T_{\max}} \right)^2.$$

Step 5 If $|\beta - \beta'| > \epsilon$, then set $\beta' = \beta$ and go to Step 2, else STOP and determine C_i as

$$C_i = f_i + \frac{\sum_L \sqrt{f_j d_j}}{\gamma\ T_{\max}} \sqrt{\frac{f_i}{d_i}}, \qquad \forall i \in L.$$

Note that there are many ways that can be adopted for setting the initial value of d_i at Step 1. Some examples are given in the following section.

6. Numerical Results and Discussions

This section gives some numerical results on the performance of the proposed algorithm in the view of optimality and computation amount, and some results on the effect of concave cost function on the link capacity assignment in packet-switched networks.

6.1 Performance of the Proposed Algorithm

First, we examine the optimality of the proposed algorithm by comparing its results with global optimal solutions. For comparison, the results obtained from the method using approximation function [8] are also given. A network with fully connected topology is considered with five thousand random traffic patterns. For the method of setting the initial value of d_i, we consider the following two methods in this paper.

method A d_i is set according to the relationship between link flow f_i and link capacity at breakpoints of the cost function as follows:

Table 1 Percentage of yielding global optimum ($p = 2$).

n	case 1	case 2	case 3	case 1′	case 2′	case 3′
3	96.0	98.8	99.1	95.7	96.6	97.8
6	91.2	98.7	98.9	90.4	95.5	96.9
10	83.9	98.2	98.5	83.0	94.1	95.7
15	77.1	97.8	98.3	74.4	93.6	95.4
21	70.7	96.8	97.9	65.0	92.7	94.4
28	61.9	94.2	97.0	54.0	90.7	93.0

Table 2 Percentage of yielding global optimum ($p = 3$).

n	case 1	case 2	case 3	case 1′	case 2′	case 3′
3	92.2	97.1	97.5	85.1	94.9	95.9
6	83.5	92.3	94.2	77.0	88.7	89.0
10	74.5	88.9	90.9	66.6	78.4	85.8
15	65.4	83.9	86.8	53.7	68.0	75.4

Table 3 Percentage of yielding global optimum ($p = 4$).

n	case 1	case 2	case 3	case 1′	case 2′	case 3′
3	92.6	98.0	98.2	85.5	93.5	94.5
6	83.9	95.3	95.9	76.1	85.5	88.8
10	75.4	92.1	93.0	67.0	77.8	82.2

$$
\begin{aligned}
d_i &= d_{0i} \text{ if } f_i < C_{0i},\\
d_i &= d_{1i} \text{ if } C_{0i} \le f_i < C_{1i},\\
&\vdots\\
d_i &= d_{(p-1)i} \text{ if } f_i \ge C_{(p-2)i}.
\end{aligned}
$$

method B d_i is set randomly among $d_{0i}, d_{1i}, \ldots,$ and $d_{(p-1)i}$.

To obtain a good solution for a CA problem, we apply the proposed algorithm and the method using approximation function in the following three cases:

- case 1 (1′): method A
- case 2 (2′): method A and 10 times of method B
- case 3 (3′): method A and 20 times of method B

In the results given below, we denote cases 1, 2, 3 for the proposed algorithm, and cases 1′, 2′, 3′ for the method using approximation function. In the cases that there are many solutions obtained, we select the best solution, i.e., the solution with the smallest network cost, as a final solution.

The relationships between the number of links in the network n and the percentage that each method yields global optimal solution are shown in Tables 1–3. Note that global optimum can be obtained by examining all link set combinations.

Next, we observe the difference between the solution obtained from each method and global optimum. Tables 4–6 show the average value of the ratio of the solution obtained from the method and global optimum.

From Tables 1–6, it can be seen that although the proposed algorithm has a lower percentage of yielding global optimum when the number of links in the network increases, its solution is very close to the global

Table 4 Average ratio of solution and global optimum ($p = 2$).

n	case 1	case 2	case 3	case 1′	case 2′	case 3′
3	1.00034	1.00012	1.00009	1.00074	1.00033	1.00030
6	1.00043	1.00004	1.00004	1.00149	1.00031	1.00029
10	1.00055	1.00005	1.00004	1.00205	1.00030	1.00028
15	1.00059	1.00004	1.00003	1.00236	1.00028	1.00026
21	1.00061	1.00004	1.00003	1.00237	1.00027	1.00025
28	1.00064	1.00004	1.00002	1.00260	1.00027	1.00025

Table 5 Average ratio of solution and global optimum ($p = 3$).

n	case 1	case 2	case 3	case 1′	case 2′	case 3′
3	1.00047	1.00018	1.00017	1.00082	1.00030	1.00026
6	1.00064	1.00030	1.00019	1.00154	1.00059	1.00056
10	1.00067	1.00025	1.00019	1.00195	1.00054	1.00046
15	1.00066	1.00024	1.00018	1.00227	1.00052	1.00042

Table 6 Average ratio of solution and global optimum ($p = 4$).

n	case 1	case 2	case 3	case 1′	case 2′	case 3′
3	1.00027	1.00008	1.00007	1.00045	1.00024	1.00020
6	1.00040	1.00014	1.00011	1.00068	1.00027	1.00021
10	1.00039	1.00012	1.00011	1.00082	1.00035	1.00027

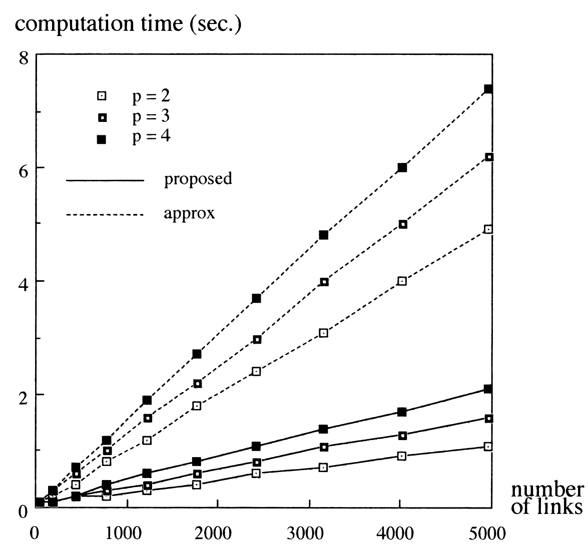


Fig. 8 Maximum computation time.

optimum. Moreover, the proposed algorithm has better performance than the method using approximation function. This means that the proposed algorithm solves the CA problem very efficiently.

Next, we investigate the computation amount of the algorithm. We then show the computation time of each method by using actual computation results. Again, a network with fully connected topology is considered, and five thousand of random traffic patterns are applied to the network. The relationships between the number of links in the network and maximum, average, and variance of computation time are shown in Figs. 8, 9, and 10, respectively (SPARC station 4 is used in determining numerical results). The values of computation time are measured for one executing time of

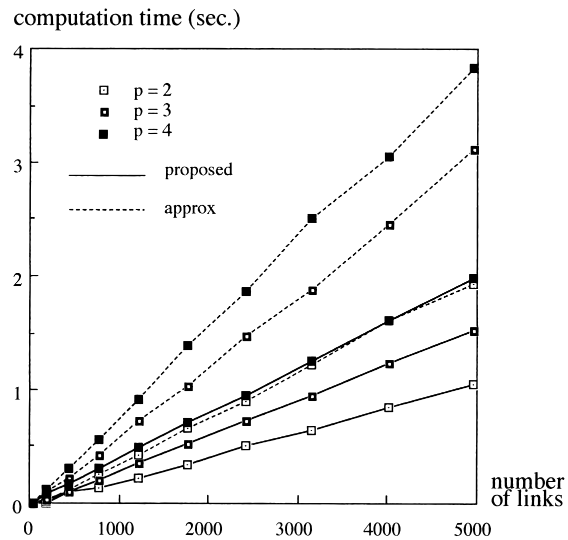


Fig. 9 Average computation time.

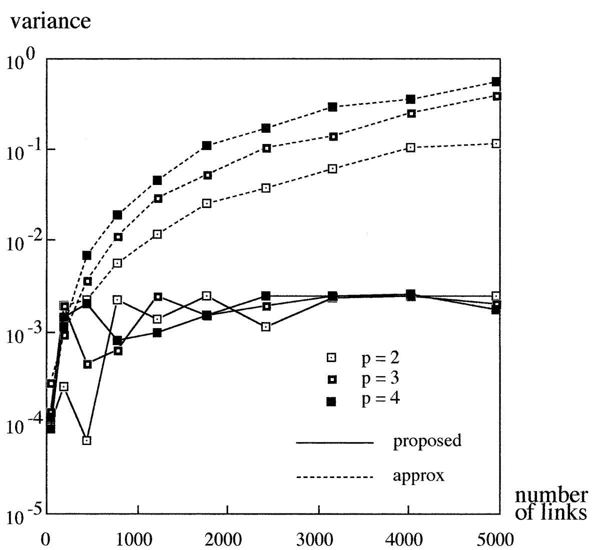


Fig. 10 Variance of computation time.

each method.

Figures 8 and 9 show that the maximum and average computation time of both methods are approximately linear with respect to the number of links in the network. (By linear regression, the values of linear correlation coefficients are very close to 1.) However, the method using approximation function needs longer computation time as can be seen from the figures.

Figure 10 shows that the variance of computation time of the proposed algorithm is very small. Hence, it can be concluded that the proposed algorithm is very practical for applying to the CA problem of a large-scale network.

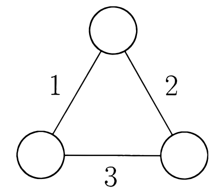


Fig. 11 A 3-node network.

6.2 Effect of Concave Link Cost Function on the Network Design

For simplicity, we focus on the case $p = 2$. A CA problem of this case is the same as the problem taking into consideration an existing network, where the per-unit cost of installing new capacity is less than that of using existing capacity. For link i, we let d_{0i} and d_{1i} be respectively the per-unit cost of existing and new capacity, and C_{0i} be the existing link capacity.

A 3-node network as shown in Fig. 11 is considered. Network parameters are set as follows: $C_{0i} = 52$ kbps for $i = 1, 2, 3$, cost function of link 1 and 2 are linear with respect to link capacity with cost coefficient (per-unit cost) equal to 1, $f_1 = f_2 = 40$ kbps, $T_{\max} = 20$ ms., mean value of packet length is 400 bits/packet. By varying f_3 and applying several pairs of (d_{03}, d_{13}), the results as shown in Fig. 12 are obtained. From Fig. 12, we can see that the capacity of link 3 (C_3) is never equal to its own existing capacity (52 kbps) for all over the entire range of f_3. Moreover, there are some values of f_3 that make the curve of link capacity value to be discontinuous. These approximated values are summarized as follows:

$$\begin{aligned}(d_{03}, d_{13}) &= (1, 0.2) \rightarrow f_3 = 29.32 \text{ kbps.},\\ (d_{03}, d_{13}) &= (1, 0.5) \rightarrow f_3 = 31.44 \text{ kbps.},\\ (d_{03}, d_{13}) &= (5, 1) \rightarrow f_3 = 37.47 \text{ kbps.},\\ (d_{03}, d_{13}) &= (10, 1) \rightarrow f_3 = 39.06 \text{ kbps.}\end{aligned}$$

At these values of f_3, the total network cost is the same for both the cases when link 3 belongs to L_0, and when it belongs to L_2. It means that we can have two solutions with the same network cost.

7. Conclusions

In this paper, we have studied the link capacity assignment problem in packet-switched networks (CA problem), where link cost function is piecewise linear concave. This type of cost function exists in many design problems, e.g., a design problem taking into consideration an existing network in long-term design, a design problem with link type selection, etc.

After formulating the CA problem, the method of link set assignment is applied, and the characteristic of the optimal solution of the problem is derived.

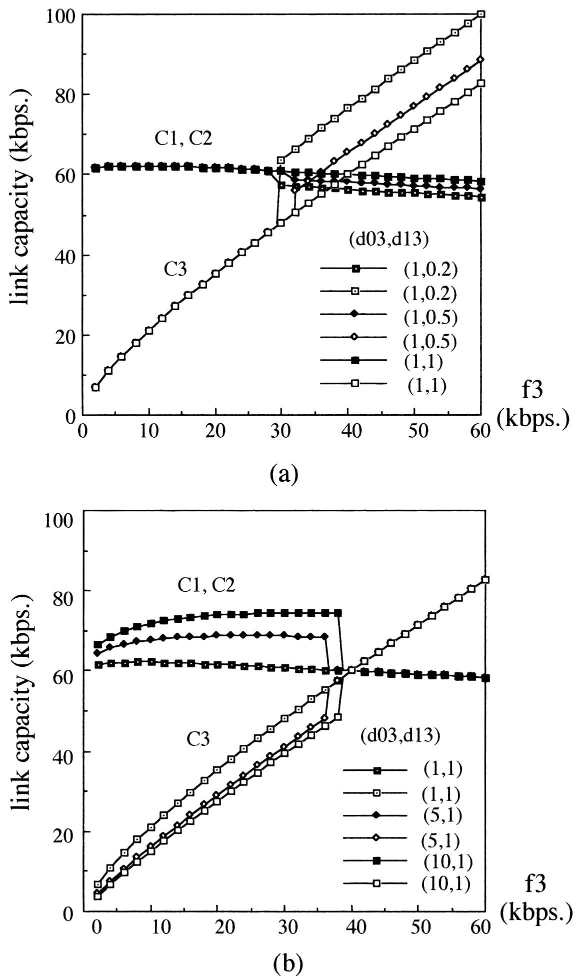


Fig. 12 Results of the 3-node network.

Based on this characteristic, it is shown that the non-differentiable link cost function can be treated as a differentiable function. As a result, the Lagrange multiplier method can be directly applied to solve the problem, and a heuristic algorithm derived from the Lagrange multiplier method is proposed.

Due to the non-convexity of the CA problem, it is hard to determine a global optimal solution. However, by investigating numerical results, it is shown that the proposed algorithm has very good performance for solving the CA problem, both from the view of optimality and computation time.

Acknowledgement

The work in this paper is supported by the Thailand Research Fund under the Post-Doctoral Research Fellowship.

References

[1] M. Gerla and L. Kleinrock, "On the topological design of distributed computer networks," IEEE Trans. Commun., vol.COM-25, no.1, pp.48–60, Jan. 1977.

[2] L. Kleinrock, Queueing Systems Volume II, Computer Applications, John Wiley & Sons, New York, 1976.

[3] S. Runggeratigul, W. Zhao, Y. Ji, A. Aizawa, and S. Asano, "Link capacity assignment in packet-switched network with existing network consideration," IEICE Trans. Commun., vol.E78-B, no.5, pp.709–719, May 1995.

[4] A. Kershenbaum, Telecommunications Network Design Algorithms, McGraw-Hill, 1993.

[5] A. Gersht and R. Weihmayer, "Joint optimization of data network design and facility selection," IEEE J. Select. Area Commun., vol.8, no.9, pp.1667–1681, Dec. 1990.

[6] B. Yaged, "Minimum cost routing for static network models," Networks, vol.1, pp.139–172, 1971.

[7] P.K. Verma, Performance Estimation of Computer Communication Networks, Computer Science Press, 1989.

[8] M. Shinohara, "Generalization of Pratt's formula for dimensioning circuit-switched network," Computer Networks and ISDN Systems, vol.20, pp.115–126, 1990.

Appendix

In this appendix, we will show that from e_{0i} defined in Sect. 4, we can conclude that if $e_{0i} = 0$, link i can be assigned to L_0 even if some of other links are changed to L_2. Also, the conclusion about e_{2i} is that, if $e_{2i} = 0$, link i can be assigned to L_2 even if some of other links are changed to L_0.

Proof: Assume that there is a link i with $e_{0i} = 0$. For this link, we have

$$C_i = f_i + \frac{b_0 + b_1 + b_0'}{\gamma\ T_{\max}} \sqrt{\frac{f_i}{d_{0i}}} < C_{0i}, \qquad (\text{A}\cdot 1)$$

or

$$\frac{1}{\sqrt{d_{0i}}} < \frac{\gamma\ T_{\max}\ (C_{0i} - f_i)}{\sqrt{f_i}\ (b_0 + b_1 + b_0')}. \qquad (\text{A}\cdot 2)$$

The definitions of b_0, b_1, and b_0' are given in Sect. 4.

Currently, all links in $L - (L_0' \cup L_2')$ are assigned to L_0 for checking link set contradiction. Let L_2'' be a set of links in $L - (L_0' \cup L_2')$ that are changed to L_2. After the change, we have b_0' becomes b^* as

$$b^* = \sum_{L_0''} \sqrt{f_j d_{0j}} + \sum_{L_2''} \sqrt{f_k d_{1k}}, \qquad (\text{A}\cdot 3)$$

where $L_0'' = L - (L_0' \cup L_2' \cup L_2'')$. See the definition of L_0' and L_2' in Sect. 4.

Note that link i under consideration must be in L_0'', since we are checking link set contradiction when it is assigned to L_0.

From $d_{0i} > d_{1i},\ \forall i \in L$, we have

$$b_0' = \sum_{L_0'' \cup L_2''} \sqrt{f_j d_{0j}} > b^*. \qquad (\text{A}\cdot 4)$$

This makes

$$\frac{\gamma\ T_{\max}\ (C_{0i} - f_i)}{\sqrt{f_i}\ (b_0 + b_1 + b_0')} < \frac{\gamma\ T_{\max}\ (C_{0i} - f_i)}{\sqrt{f_i}\ (b_0 + b_1 + b^*)}. \qquad (\text{A}\cdot 5)$$

From Eqs. (A$\cdot$2) and (A$\cdot$5),

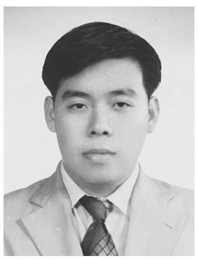
$$\frac{1}{\sqrt{d_{0i}}} < \frac{\gamma\, T_{\max}\, (C_{0i} - f_i)}{\sqrt{f_i}\, (b_0 + b_1 + b^*)}, \qquad (\mathrm{A}\cdot 6)$$

or

$$C_i = f_i + \frac{b_0 + b_1 + b^*}{\gamma\, T_{\max}} \sqrt{\frac{f_i}{d_{0i}}} < C_{0i}, \qquad (\mathrm{A}\cdot 7)$$

which implies that e_{0i} is still equal to 0 although link set combination is changed. In other words, link i with $e_{0i} = 0$ can belong to L_0 without any contradiction.

In the same manner, we can also prove that link i with $e_{2i} = 0$ can be assigned to L_2 without any contradiction. □



Suwan Runggeratigul received the B.E. degree in electrical engineering from Chulalongkorn University, Thailand in 1989, and the M.E. and D.Eng. degrees in electronic engineering from the University of Tokyo, Japan in 1992, and 1995, respectively. He was a visiting researcher at the National Center for Science Information Systems, and Nippon Telegraph and Telephone Multimedia Networks Laboratory, Japan from 1995 to 1997. He is now with Sirindhorn International Institute of Technology, Thammasat University, Thailand. His research interests include analysis of communication systems and network design.

Sawasd Tantaratana was born in Thailand. He received his B.E.E. degree from the University of Minnesota in 1971, M.S. degree from Stanford University in 1972, and Ph.D. degree from Princeton University in 1977, all in electrical engineering. He has spent most of his career in teaching in both Thai and U.S. universities: King Mongkut's Institute of Technology, Thonburi, Thailand (1972–1974 and 1977–1980); Auburn University, Alabama, USA (1981–1984); University of Massachusetts at Amherst, Massachusetts, USA (1986–1996); and Sirindhorn International Institute of Technology, Thammasat University, Pathumthani, Thailand (since 1997), where he is a Professor of Electrical Engineering. He was a visiting faculty member at the University of Illinois at Urbana-Champaign, Illinois, USA, from 1980 to 1981, a member of the Technical Staff at AT&T Bell Laboratories in Holmdel, New Jersey, USA, from 1984 to 1986, and a senior researcher at the National Electronic and Computer Technology Center, Bangkok, Thailand, from 1996–1997. His research interests are signal processing and communications.