2005 American Control Conference
June 8-10, 2005. Portland, OR, USA

Power Control in Lognormal Fading Wireless Channels with Uptime Probability Specifications via Robust Geometric Programming

Kan-Lin Hsiung, Seung-Jean Kim and Stephen Boyd
Information Systems Laboratory
Department of Electrical Engineering, Stanford University
{klhsiung,sjkim,boyd}@stanford.edu

Abstract— **This paper concerns power control in lognormal fading wireless channels with correlated interference. We consider an outage-based quality of service (QoS) specification, which requires the uptime probability, *i.e.*, the probability that no transmitter/receiver pair experiences any outage due to fading, is kept above a given level. The problem of finding an optimal power allocation to achieve this QoS goal over *lognormal fading* wireless channels can be posed as a stochastic geometric program (GP) with joint probabilistic constraints. This stochastic geometric program is extremely hard to solve in general, compared with the stochastic GP associated with Rayleigh fading channels. In this paper, we describe a suboptimal approach based on recently proposed robust geometric programming. With a good compromise between computational efficiency and accuracy, this robust GP relaxation approach finds a power allocation which meets the QoS requirement on the uptime probability. A numerical example is given to demonstrate the method.**

I. INTRODUCTION

A common QoS requirement in a modern cellular system is that the outage probability, *i.e.*, the probability of the signal-to-interference ratio (SIR) or signal-to-interference-plus-noise ratio (SINR) being below some threshold, is kept below a specified level. Power control provides an intelligent way of determining transmitting power to achieve the outage probability specifications [1], [2]. In many other aspects, power control has also be shown to be an effective way to improve the performance of wireless communication systems. For example, careful allocation of power can increase utilization in interference-limited systems (such as CDMA) or multiple access systems with frequency-reuse schemes (*e.g.*, in FDMA). It is also effective in mitigating near-far effects occurring when the signal received by a remote user is attenuated significantly due to disturbing from a nearby interferer. In addition, power allocation is critical in wireless networks for longer battery life: by allocating minimum power across all mobile devices, users only need to expend sufficient power for acceptable QoS.

For medium-scale fading (commonly referred to as *shadowing*), the underlying signal power can be well described by lognormal statistics, *i.e.*, the signal measured in decibel or nats has a normal distribution (see, *e.g.*, [3]). In addition, in some situations shadowing of the desired and interfering signals is the main source of performance degradation in the presence of channel interference. For instance, in wireless systems employing diversity techniques to combat effects of multipath (short-term) fading, the co-channel interference considerations are more strongly dependent on the large scale signal variation due to shadowing [4]. For these reasons, it is beneficial to take into account the effects of lognormal fading in designing cellular systems.

The effect of lognormal fadings on performance of wireless communication systems has been a topic of intensive research. Although a lot of work has assumed that fadings are statistically independent, they may be correlated. For instance, the signals that a mobile station receives from surrounding base stations may be shadowed by the same obstacles in the neighborhood of the receiver; correlation between shadowings has been experimentally found [5], [6].

In the presence of lognormal channel interference, computing the outage probability (or its bounds) often involves calculating the mean and variance of the sum of lognormal random variables (see, *e.g.*, [7], [8], [9], [10] and references therein). A general closed-form expression (in terms of known functions) for the lognormal sum distribution is not available (even for the case of independent lognormal random variables). Several approximative methods have been suggested in the literature to compute both the outage probability and the underlying lognormal sum distribution (see, *e.g.*, [10] and references therein). Since a good approximation for the probability of outage with lognormal shadowed interferers is in general a highly nonlinear function of the transmitter powers, only heuristics or local optimization algorithms can be used to find (local) solutions to

0-7803-9098-9/05/$25.00 ©2005 AACC

the problem of minimizing the total transmitter power subject to outage probability constraints, *e.g.*, [11], [12]. This is in sharp contrast with some short-term fading environments, *e.g.*, Rayleigh, in which the probability of outage with (independent) Rayleigh faded interferers can be expressed analytically as a *posynomial* function of transmitter powers and hence geometric programming can be used to achieve power minimization with outage probability specifications [1].

This paper concerns power control in lognormal fading wireless channels with *correlated* interference. The statistical variation and correlation of both the received signal and interference power are taken into account explicitly. We introduce an outage-based QoS specification, which requires that the SINR of *each* transmitter/receiver pair keeps above a given threshold for highly probable channel conditions (determined by the underlying lognormal statistics and QoS threshold). This measure can be interpreted as the fraction of time no transmitter/receiver pair experiences any outage due to fading. The problem of finding an optimal power allocation to achieve this QoS goal over lognormal fading wireless channels can be posed as a stochastic geometric program with joint probabilistic constraints. This stochastic GP is extremely hard to solve, unlike the stochastic geometric program associated with Rayleigh fading channels. In this paper, we describe a suboptimal approach based on recently proposed robust geometric programming [13]. With a good compromise between computational efficiency and accuracy, this robust GP relaxation approach finds a power allocation which meets the QoS requirement on the uptime probability. A numerical example is given to demonstrate the method.

II. BACKGROUND

A. Geometric Programming

The convex function $\mathbf{lse} : \mathbb{R}^k \to \mathbb{R}$, defined as

$$\mathbf{lse}(y_1, \ldots, y_k) = \log(e^{y_1} + \cdots + e^{y_k}), \quad (1)$$

is called the *log-sum-exp* function.

A *geometric program* (in convex form) can be formulated as

$$\begin{array}{ll} \text{minimize} & \mathbf{lse}(A_0 y + b_0) \\ \text{subject to} & \mathbf{lse}(A_s y + b_s) \le 0, \quad s = 1, \ldots, m, \\ & Hy + h = 0, \end{array} \quad (2)$$

where $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ are the optimization variables, and $A_s \in \mathbb{R}^{K_s \times n}$, $b_s \in \mathbb{R}^{K_s}$, $s = 0, 1, \ldots, m$, $H \in \mathbb{R}^{l \times n}$, $h \in \mathbb{R}^l$ are the problem data. (For more details about GP and its applications, see [14], [15].)

B. Robust GP with Ellipsoidal Uncertainty

Assume (A_s, b_s) in (2) are uncertain, but known to belong to the image of a set $\mathcal{U} \subset \mathbb{R}^L$ under an affine mapping:

$$(\tilde{A}_s(u), \tilde{b}_s(u)) = \left(A_s^0 + \sum_{j=1}^{L} u_j A_s^j,\ b_s^0 + \sum_{j=1}^{L} u_j b_s^j \right), \quad (3)$$

where $A_s^j \in \mathbb{R}^{K_s \times n}$ and $b_s^j \in \mathbb{R}^{K_s}$. To simplify notations, we define $f_s : \mathbb{R}^n \times \mathbb{R}^L \to \mathbb{R}$ as

$$f_s(y, u) = \mathbf{lse}(\tilde{A}_s(u) y + \tilde{b}_s(u)) \quad (4)$$

and $f_0 : \mathbb{R}^n \to \mathbb{R}$ as

$$f_0(y) = \mathbf{lse}(A_0 y + b_0). \quad (5)$$

The corresponding robust GP (in convex form) can then be formulated as

$$\begin{array}{ll} \text{minimize} & f_0(y) \\ \text{subject to} & \sup_{u \in \mathcal{U}} f_s(y, u) \le 0,\ s = 1, \ldots, m, \\ & Hy + g = 0, \end{array} \quad (6)$$

where $H \in \mathbb{R}^{l \times n}$, $g \in \mathbb{R}^l$. In addition, we assume the robust GP (6) has *ellipsoidal uncertainty*, in which $\mathcal{U}$ is an ellipsoid:

$$\mathcal{U} = \{\bar{u} + P\rho \mid \|\rho\|_2 \le 1,\ \rho \in \mathbb{R}^L\}, \quad (7)$$

where $\bar{u} \in \mathbb{R}^L$ and $P \in \mathbb{R}^{L \times L}$. An approximation method with modest computational complexity has been proposed for the robust GP (6) with ellipsoidal uncertainty (7); see [13] for more details.

III. CHANNEL MODEL AND QoS METRICS

A. Lognormal Fading Channels

In this paper the following setup for lognormal fading channels is considered. We have n transmitters, labelled $1, \ldots, n$, which transmit at (positive) power levels $P_1, \ldots, P_n$ respectively. We also have n receivers, labelled $1, \ldots, n$; receiver i is meant to receive the signal from transmitter i. The power received from transmitter j, at receiver i, is given by $F_{ij} G_{ij} P_j$, where $G_{ij} > 0$ represents the path gain (not including fading) from transmitter j to receiver i. In the analysis below, we assume that G_{ij} are constant, *i.e.*, do not change (much) with time.

To model lognormal fading over wireless channels, F_{ij} are assumed to be lognormal random variables:

$$F_{ij} \sim LN(\mu_{ij}, \sigma_{ij}^2), \quad i, j = 1, \ldots, n. \quad (8)$$

(Recall that a random variable x has the lognormal distribution $x \sim LN(\mu, \sigma^2)$ if its probability density function has the form:

$$p_x(\xi) = \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{\xi}e^{-(\log \xi - \mu)^2/(2\sigma^2)}, \quad (9)$$

where $0 < \xi < \infty$, $-\infty < \mu < \infty$, and $\sigma > 0$.) Note that here we do not pose the assumption that F_{ij} are independent. Wireless channels over which both desired signals and interference signals are subject to lognormal fading are referred to as *lognormal fading channels*.

B. Uptime Probability

At the ith receiver, the signal power is $F_{ii}G_{ii}P_i$, and the total interference power is given by $\sum_{k \neq i} F_{ik}G_{ik}P_k$. The *signal-to-interference-plus-noise ratio* (SINR) of the ith receiver/transmitter pair is given by

$$S_i = \frac{F_{ii}G_{ii}P_i}{\beta_i + \sum_{k \neq i} F_{ik}G_{ik}P_k}, \quad (10)$$

where β_i represents the noise power in the ith receiver. In this paper we assume β_i are constant.

One common QoS requirement is that SINR of the ith receiver/transmitter pair must be kept above a given threshold $S_i^{\min}$:

$$S_i \geq S_i^{\min}, \quad i = 1, \ldots, n.$$

The outage probability of the ith receiver/transmitter pair is therefore given by

$$\mathcal{O}_i(P_1, \ldots, P_n) = \mathbf{Pr}(S_i < S_i^{\min}), \quad i = 1, \ldots, n. \quad (11)$$

Here and elsewhere $\mathbf{Pr}(A)$ is the probability of event A. Note that in a lognormal fading environment, S_i is a random variable with what would appear to be a very complex distribution, since it is the ratio of a lognormal random variable to a sum of lognormal random variables (with different means). The analytical expression for its density is yet unknown.

Assume the required QoS concerns the 'integrated' performance of n receiver/transmitter pairs (in contrast to posing (outage) probability constraints individually as in (11)); we define the *uptime probability* as

$$\mathcal{J}(P_1, \ldots, P_n) = \mathbf{Pr}(S_i \geq S_i^{\min}, \ i = 1, \ldots, n), \quad (12)$$

which can be interpreted as the fraction of time when no transmitter/receiver pair experiences any outage due to fading.

Note that in our expression for $\mathcal{J}$, statistical variation of both received signal power and received interference power is taken into account. In this paper, the requested QoS is provided when the uptime probability $\mathcal{J}$ exceeds a given threshold.

IV. STOCHASTIC GP WITH JOINT PROBABILISTIC CONSTRAINTS

A. Formulation

In the robust GP (6), we assume $u \in \mathbb{R}^L$ is deterministic but uncertain. In some cases, however, $u \in \mathbb{R}^L$ is a random vector with given distribution. Accordingly, we can define the following stochastic GP:

$$\begin{array}{ll} \text{minimize} & f_0(y) \\ \text{subject to} & \mathbf{Pr}(f_s(y,u) \leq 0, \ s = 1, \ldots, m) \geq \alpha, \\ & Hy + g = 0, \end{array} \quad (13)$$

where $H \in \mathbb{R}^{l \times n}$, $g \in \mathbb{R}^l$, and $0 < \alpha < 1$ is a given threshold. If $y \in \mathbb{R}^n$ is feasible to the above stochastic GP, we say that the *risk of loss* of y is less than $1 - \alpha$.

Note that the stochastic GP (13) is an example of optimization with a *joint probabilistic constraint*, which is different from imposing the constraints individually, *i.e.*

$$\mathbf{Pr}(f_s(y,u) \leq 0) \geq \alpha_s, \quad s = 1, \ldots, m,$$

where $\alpha_s > 0$ are thresholds for individual probabilistic constraints. In general, optimization with joint probabilistic constraints is very difficult to handle, both theoretically and computationally [16]. In §IV-B we will show that a relaxation method via the robust GP (6) with the ellipsoid uncertainty (7) can be used to find feasible solutions of the stochastic GP (13), provided that $u \in \mathbb{R}^L$ is jointly normal with given covariance matrix.

B. Robust GP Relaxation

Recall that a normal random variable $u \in \mathbb{R}^n$ with mean $\bar{u}$ and covariance matrix $\Sigma = \Sigma^T > 0$, denoted by $u \sim N(\bar{u}, \Sigma)$, has the probability density function

$$p_u(\xi) = (2\pi)^{-n/2}(\det \Sigma)^{-1/2}e^{-1/2(\xi - \bar{u})^T\Sigma^{-1}(\xi - \bar{u})}. \quad (14)$$

Obviously $p_u(\xi)$ is constant for $(\xi - \bar{u})^T\Sigma^{-1}(\xi - \bar{u}) = \gamma$, *i.e.*, on the surface of the ellipsoid

$$\mathcal{E}_\gamma = \{\xi \in \mathbb{R}^n \mid (\xi - \bar{u})^T\Sigma^{-1}(\xi - \bar{u}) \leq \gamma\}. \quad (15)$$

Here $\mathcal{E}_\gamma$ is called a *confidence ellipsoid* of u. It is well-known that the nonnegative random variable

$$(u - \bar{u})^T\Sigma^{-1}(u - \bar{u})$$

has a chi-squared distribution with degree n, *i.e.*,

$$\mathbf{Pr}(u \in \mathcal{E}_\gamma) = F_{\chi_n^2}(\gamma), \quad (16)$$

where $F_{\chi_n^2}$ is the cumulated distribution function of χ_n^2. (See, *e.g.*, [17].)

Now we suppose that $u \in \mathbb{R}^L$ in the stochastic GP (13) is normally distributed with the density function (14). In this case, feasible solutions of (13) can be obtained via robust GP relaxation as follows. Define the uncertainty set $\mathcal{U}$ as

$$\mathcal{U} = \mathcal{E}_\gamma, \tag{17}$$

where $\mathcal{E}_\gamma$ is the confidence ellipsoid define in (15), and γ is uniquely determined by

$$F_{\chi_n^2}(\gamma) = \alpha. \tag{18}$$

It is straightforward to see that all the feasible solutions of the robust GP (6) with the ellipsoidal uncertainty (17) are feasible to the stochastic GP (13) with jointly normal $u \in \mathbb{R}^L$, *i.e.*, risk of loss is guaranteed to be less than $1 - \alpha$.

V. POWER ALLOCATION WITH UPTIME PROBABILITY CONSTRAINTS

A. Power Minimization as Stochastic GP

The problem of minimizing the total transmitter power, subject to a lower bound on the probability that each transmitter/receiver pair has no outage due to fading, can be formulated as

$$\begin{array}{ll} \text{minimize} & \sum_{i=1}^n P_i \\ \text{subject to} & \mathcal{J}(P_1, \ldots, P_n) \geq \alpha, \end{array} \tag{19}$$

where $\mathcal{J}$ is the uptime probability defined in (12) with the lognormal random variables F_{ij} introduced in (8), and $0 < \alpha < 1$ is the minimum allowed uptime probability.

Since a lognormal random variable $v \sim LN(\mu, \sigma^2)$ is associated with a standard normal variable $p \sim N(0,1)$ through $v = e^{\mu + \sigma p}$, the fading factors F_{ij} can be expressed as

$$F_{ij} = \exp(\mu_{ij} + \sigma_{ij} p_{ij}), \quad i, j = 1, \ldots, n \tag{20}$$

with $p_{ij} \sim N(0,1)$. For $i, j = 1, \ldots, n$, define

$$\begin{aligned} b_i &= \log \beta_i, \quad d_i = \log S_i^{\min}, \\ y_j &= \log P_j, \quad c_{ij} = \log G_{ij}. \end{aligned}$$

Hence by (20), $S_i^{\min} S_i^{-1}$ can be reformulated as a sum of exponentials:

$$\begin{aligned} & S_i^{\min}\left(\beta_i + \sum_{k \neq i} F_{ik} G_{ik} P_k\right)(F_{ii} G_{ii} P_i)^{-1} \\ = \;& \exp\left[(b_i + d_i - \mu_{ii} - c_{ii}) - \sigma_{ii} p_{ii} - y_i \right] \\ + & \sum_{k \neq i} \exp\left[(d_i + (c_{ik} - c_{ii}) + (\mu_{ik} - \mu_{ii})) \right. \\ & \qquad \left. + (\sigma_{ik} p_{ik} - \sigma_{ii} p_{ii}) + (y_k - y_i) \right], \end{aligned}$$

where $p_{ij} \sim N(0,1)$, $i, j = 1, \ldots, n$. Consequently, given the lognormal distributions (8), we can easily find $(A_s^j, b_s^j) \in \mathbb{R}^{n \times n} \times \mathbb{R}^n$, $s = 1, \ldots, n$, $j = 1, \ldots, n^2$ such that the uptime probability constraint in (19), *i.e.*, $\mathcal{J}(P_1, \ldots, P_n) \geq \alpha$, can be reformulated in the form of

$$\mathbf{Pr}(f_s(y,u) \leq 0, \; s = 1, \ldots, n) \geq \alpha,$$

where $f_s(y,u)$ is defined in (4), and $u = (u_1, \ldots, u_{n^2})$ with

$$u_i \sim N(0,1), \quad i = 1, \ldots, n^2. \tag{21}$$

The power minimization (19) can therefore be cast in form of the stochastic GP (13).

In summary, the problem of minimizing the total transmitter power with the uptime probability constraint over lognormal fading channels can be expressed in form of the stochastic GP with a joint probabilistic constraint as defined in (13).

B. Power Allocation via Robust GP Relaxation

Assume $\Sigma_u = \Sigma_u^T > 0$ is the covariance matrix of the random vector $u \in \mathbb{R}^{n^2}$ introduced in (21). (Obviously, $\Sigma_u = I_{n^2}$ if the fading factors F_{ij}, $i, j = 1, \ldots, n$ are independent.)

Feasible solutions of the power minimization (19) can be obtained by the robust GP relaxation described in §IV-B as follows. Define the ellipsoidal uncertainty set

$$\mathcal{U} = \{\xi \in \mathbb{R}^{n^2} \mid \xi^T \Sigma_u^{-1} \xi \leq \gamma\}$$

for some γ, which is uniquely determined by $F_{\chi_n^2}(\gamma) = \alpha$. Here α is the QoS threshold assigned in (19). Then any feasible solution $(\bar{y}_1, \ldots, \bar{y}_n)$ of the robust GP (6) with the above ellipsoidal uncertainty $\mathcal{U}$ is feasible to the stochastic GP of the form (13) that is reformulated from the power minimization (19) as discussed in §V-A. The power allocation

$$(P_1, \ldots, P_n) = (e^{\bar{y}_1}, \ldots, e^{\bar{y}_n})$$

can therefore achieve the QoS requirement on the uptime probability, *i.e.*, $\mathcal{J}(P_1, \ldots, P_n) \geq \alpha$.

VI. SIMULATION RESULTS

In this section, we give a numerical example to demonstrate the power allocation method described thus far. We consider a system with 6 transmitters and receivers with lognormal fading. All the path gains G_{ii} (from ith transmitter to ith receiver) are assumed to be one. The cross gains G_{ij}, $i \neq j$ are generated as independent random variables uniformly distributed in the interval $[0, 0.01]$. All the noise powers β_i are assumed to be 0.01. The fading factors F_{ij} are lognormal variables with mean 0.5 and variance $(0.15)^2$.

The correlation between the standard normal random variables $u_1, \ldots, u_{36}$ introduced in (21) is described by the covariance matrix $\Sigma \in \mathbb{R}^{36\times 36}$ with diagonal entries 1 and off-diagonal entries 0.1.

We consider the QoS thresholds $\alpha = 0.9$, 0.8, 0.7 and 0.6 respectively. For each of the four QoS thresholds, we vary the SINR thresholds $S_i^{\min}$ (which are assumed to be the same for each transmitter/receiver pair) from 3 to 15. For each value of $S_i^{\min}$, we generate 20 instances of (19), in which the cross gains are randomly selected as described above. Then we compute the total power $P_1 + \cdots + P_6$ for each instance by the proposed robust GP relaxation approach, and the average of the 20 instances is used to obtain each data point in Fig. 1, which shows the tradeoff between the total transmitter power and SINR threshold.

For fixed QoS threshold α, the total power increases as higher SINR threshold $S_i^{\min}$ is assigned. For high QoS threshold, *e.g.*, $\alpha = 0.9$, the total power increases drastically as SINR threshold increases. This implies that the 'price of robustness' is high in this example.

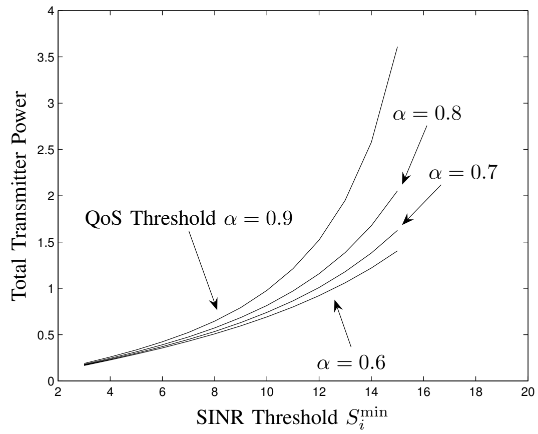

Fig. 1. Total transmitter power versus SINR threshold for a system with 6 wireless links. For high QoS threshold, *e.g.*, $\alpha = 0.9$, the total power increases drastically in SINR threshold.

VII. CONCLUSIONS

In this paper, we have described a power control scheme in lognormal fading wireless channels with correlated interference. With a good compromise between computational complexity and performance, this robust GP relaxation method finds a power allocation that meets the given QoS requirement on the uptime probability. A natural question arises on the performance loss of this suboptimal power allocation scheme. Since the exact optimal solution of the original problem is very hard to find, it is very difficult to answer this question. Our preliminary study based on Monte Carlo analysis suggests that the performance loss is not significant in most cases.

VIII. ACKNOWLEDGMENTS

This work was supported in part by the MARCO Focus Center for Circuit & System Solutions (C2S2, www.c2s2.org), under contract 2003-CT-888. We thank anonymous reviewers for their helpful comments and suggestions.

REFERENCES

[1] S. Kandukuri and S. Boyd. Optimal power control in interference-limited fading wireless channels with outage-probability specifications. *IEEE Trans. on Wireless Communications*, 1(1):46–55, January 2002.

[2] J. Papandriopoulos, J. S. Evans, and S. Dey. Outage-based power control for generalized multiuser fading channels. In *Proc. IEEE ICC'04*, volume 1, pages 327–331, Paris, France, June 2004.

[3] W. C. Jakes, editor. *Microwave Mobile Communications*. IEEE Press, Piscataway, NJ, 1994.

[4] D. Cox. Cochannel interference considerations in frequency reuse small-coverage-area radio systems. *IEEE Trans. Communications*, 30(1):135–142, January 1982.

[5] V. Graziano. Propagation correlations at 900 MHz. *IEEE Trans. Vehicular Technology*, 27:182–189, November 1978.

[6] F. Graziosi, M. Pratesi, M. Ruggieri, and F. Santucci. A multi-cell model of handover initiation in mobile cellular networks. *IEEE Trans. Vehicular Technology*, 48(3):802–814, May 1999.

[7] A. A. Abu-Dayya and N. C. Beaulieu. Outage probabilities in the presence of correlated lognormal interferers. *IEEE Trans. Vehicular Technology*, 43(1):164–173, February 1994.

[8] A. Ligeti. Outage probability in the presence of correlated lognormal useful and interfering components. *IEEE Communications Letters*, 4(1):15–17, January 2000.

[9] F. Berggren and S. B. Slimane. A simple bound on the outage probability with lognormal interferers. *IEEE Communications Letters*, 8(5):271–273, May 2004.

[10] G. L. Stüber. *Principles of Mobile Commnication*. Kluwer Academic, 2nd edition, 2001.

[11] T. Heikkinen. A model for stochastic power control under log-normal distribution. The 7th Canadian Workshop on Information Theory (Session 4: Wireless Systems), June 2001. University of British Columbia, Vancouver, BC.

[12] M. O. Hasna and M.-S. Alouini. Optimum power allocation for soft handoff algorithms over lognormal shadowing channels. In *Proc. IEEE ICC'02*, volume 5, pages 3207–3211, 2002.

[13] K.-L. Hsiung, S.-J. Kim, and S. Boyd. Robust geometric programming via piecewise linear approximation. Revised for publication in *Mathematical Programming*, March 2005. Available at `http://www.stanford.edu/~boyd/rgp.html`.

[14] S. Boyd, S. J. Kim, L. Vandenberghe, and A. Hassibi. A tutorial on geometric programming. Technical report, Electrical Engineering Department, Stanford University, September 2004. Available at `www.stanford.edu/~boyd/gp_tutorial.html`.

[15] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge University Press, 2004.

[16] A. Ruszczynski and A. Shapiro, editors. *Stochastic Programming*. Handbook in Operations Research and Management Science. Elsevier, 2003.

[17] K. V. Mardia, J. T. Kent, and J. M. Bibby. *Multivariate Analysis*. Academic Press, London, 1979.