Semidefinite Programming Relaxation Approach for Multiuser Detection of QAM Signals

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***Abstract*— A semidefinite programming (SDP) relaxation approach is proposed to solve multiuser detection problems in systems with M-ary quadrature amplitude modulation (M-QAM). In the proposed approach, the optimal M-ary maximum likelihood (ML) detection is carried out by converting the associated M-ary integer programming problem into a binary integer programming problem. Then a relaxation approach is adopted to convert the binary integer programming problem into an SDP problem. This relaxation process leads to a detector of much reduced complexity. A multistage approach is then proposed to improve the performance of the SDP relaxation based detectors. Computer simulations demonstrate that the symbol-error rate (SER) performance offered by the proposed multistage SDP relaxation based detectors outperforms that of several existing suboptimal detectors.**

***Index Terms*— Semidefinite programming, multistage detection, multiuser detection, QAM.**

I. Introduction

MULTIUSER detection is a useful technique that has been used in many applications such as code-division multiple-access (CDMA) multiuser communication systems, single-user multiple-input multiple-output (MIMO) communication systems, etc. Semidefinite programming (SDP) approach is an important suboptimal multiuser detection method initially proposed to detect binary or quadrature phase shift keying (BPSK/QPSK) signals. In multiuser detectors based on SDP approach, the optimal maximum likelihood (ML) detection problem is carried out by 'relaxing' the associated combinatorial programming problem into an SDP problem with both the objective function and the constraint functions being convex functions of continuous variables. Multiuser detector based on SDP approach has been shown to yield near-optimal detection performance in detecting BPSK/QPSK signals [1]-[4].

In modern wideband wireless communication systems, however, high-order modulation methods, such as M-ary phase shift keying (M-PSK) modulation or M-ary quadrature amplitude modulation (M-QAM), are often adopted. Therefore, there has been much interest recently in extending the SDP approach to detect high-order modulated signals. In [5], SDP approach was extended to the detection of general M-PSK signals, and in [6], SDP approach was applied to detect 16-QAM signals. Although it is possible to extend the polynomial relaxation in [6] to approximate ML detection for more general M-QAM, the resulting optimization problem would be highly complicated when the constellation is large.

In this paper, a new multiuser detector based on SDP approach is proposed for systems with general rectangular M-QAM. In the proposed detector, the variable associated with each M-QAM symbol is mapped into a sequence of binary variables. Therefore, the M-ary combinatorial problem associated with the optimal M-QAM ML detection is converted into a binary combinatorial problem, which is then relaxed to an SDP problem.

Although the M-ary combinatorial problem associated with the optimal M-QAM ML detection can be solved efficiently by SDP relaxation methods, simulation results show that when M is relatively large, e.g. $M \geq 64$, the performance obtained by using the SDP relaxation methods is not near-optimal.

It is noted that the detection error probabilities of the sequence of binary variables associated with each QAM symbol are unbalanced, i.e., some of the binary variables are more robust to detection errors than others. For this reason, it would be desirable to detect those binary variables which are more robust to detection errors first. Motivated by this observation, a multistage approach can be adopted to improve the performance of the SDP relaxation based detector. In the multistage approach, decisions are made only for those binary variables which can be detected with higher accuracy in each stage. Based on these decisions, the original problem is reduced to a smaller-sized SDP problem for the undetermined binary variables. This process is continued until all binary variables are determined.

The rest of this paper is organized as follows. In Section II, a signal model for systems with M-QAM and the associated optimal ML detection problem are described. In Section III, the SDP relaxation to the M-QAM detection problem is discussed. The multistage approach is proposed in Section IV for performance improvement. Simulation results are presented to compare the performance of the proposed SDP relaxation based detectors with other multiuser detectors in terms of symbol-error rate (SER) in Section V. Conclusions are made in Section VI.

II. Signal Model

In this paper, a signal model for an MIMO communication system using M-QAM is considered. The transmitted symbols are denoted as $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_K]^T$. Each symbol x_k ($k = 1, \ \ldots, \ K$) is taken, independently and with equal probability, from a complex alphabet set $\mathcal{A}$ of cardinality M,

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i.e., $x_k \in \mathcal{A} = \{I_m + jQ_n\}$ with I_m ($m = 1, \ldots, \hat{M}$) and Q_n ($n = 1, \ldots, \hat{N}$) representing the values of the In-phase (I) and Quadrature (Q) components, respectively and $\hat{M}\hat{N} = M$ [7]. The MIMO channel is described by an $L \times K$ ($K \leq L$) complex matrix, which is denoted as $\mathbf{C}$. Note that the MIMO channel here is taken in a broad context: it may be the real physical channel as in a layered MIMO communication system or simply the collection of user signature signals in a synchronous multiuser communication system over a Gaussian channel. The received signal, $\mathbf{r} = [r_1\ r_2\ \cdots\ r_L]^T$, is then given by

$$\mathbf{r} = \mathbf{C}\mathbf{x} + \boldsymbol{\eta} \tag{1}$$

where $\boldsymbol{\eta} = [\eta_1\ \eta_2\ \cdots\ \eta_L]^T$ is a noise vector of zero-mean complex Gaussian variables whose covariance matrix is given by $2\sigma^2\mathbf{I}_L$ with $\mathbf{I}_L$ being an $L \times L$ identity matrix.

The objective of multiuser detection is to recover $\mathbf{x}$ from $\mathbf{r}$ in (1). It is shown that the ML detection is carried out by solving the following optimization problem [8]

$$\text{minimize} \quad \mathbf{x}^H\mathbf{H}\mathbf{x} + \text{Re}\{\mathbf{x}^H\mathbf{p}\} \tag{2a}$$

$$\text{subject to} : x_i \in \mathcal{A} = \{I_m + jQ_n\} \text{ for } i = 1, \ldots, K; \quad m = 1, \ldots, \hat{M};\ n = 1, \ldots, \hat{N} \tag{2b}$$

where $\mathbf{H} = \mathbf{C}^H\mathbf{C}$, $\mathbf{p} = -2\mathbf{C}^H\mathbf{r}$ and x_i denotes the ith component of $\mathbf{x}$.

The feasible region of the problem in (2) is composed of all constellation points in the alphabet set $\mathcal{A}$. Because of the constraints in (2b), the optimization problem in (2) is a combinatorial problem in nature. In general, an exhaustive search is required to obtain the solution to the problem (2), in which the computational complexity involved is $\mathcal{O}(M^K)$. Therefore, the implementation of the ML detector based on exhaustive search is prohibitively high even for a system with small number of simultaneous transmitted symbols.

III. Semidefinite Programming Relaxation for M-QAM ML Detection

It can be shown that the problem in (2) is equivalent to the following one that only involves real-valued variables, i.e.,

$$\text{minimize} \quad \tilde{\mathbf{x}}^T\tilde{\mathbf{H}}\tilde{\mathbf{x}} + \tilde{\mathbf{x}}^T\tilde{\mathbf{p}} \tag{3a}$$

$$\text{subject to} : \tilde{x}_i \in \mathcal{B}_R \quad \text{for } i = 1, \ldots, K \tag{3b}$$

$$\tilde{x}_i \in \mathcal{B}_I \quad \text{for } i = K+1, \ldots, 2K \tag{3c}$$

where

$$\tilde{\mathbf{H}} = \begin{bmatrix} \text{Re}\{\mathbf{H}\} & -\text{Im}\{\mathbf{H}\} \\ \text{Im}\{\mathbf{H}\} & \text{Re}\{\mathbf{H}\} \end{bmatrix}$$

$\tilde{\mathbf{x}} = [\text{Re}\{\mathbf{x}^T\}\ \text{Im}\{\mathbf{x}^T\}]^T$, $\tilde{\mathbf{p}} = [\text{Re}\{\mathbf{p}^T\}\ \text{Im}\{\mathbf{p}^T\}]^T$, $\tilde{x}_i$ denotes the ith component of $\tilde{\mathbf{x}}$, $\mathcal{B}_R$ and $\mathcal{B}_I$ represent the sets containing all real values that I- and Q-components can take, respectively.

In what follows, the detection problem in a system using M-QAM with rectangular constellations will be considered. For any symbol drawn from a rectangular M-QAM constellation, both of its real and imaginary components can be represented as

$$\tilde{x}_i = \sum_{n=1}^{N} \omega_{n,i}\alpha_{n,i} \quad \text{for } i = 1, \ldots, 2K \tag{4}$$

TABLE I
Summary of SDP Relaxation For M-QAM Multiuser Detection

8-QAM	$\tilde{\mathbf{x}} = \mathbf{T}_2^T\mathbf{y}$ $\mathbf{y} = [\boldsymbol{\alpha}_1^T\ \boldsymbol{\alpha}_2^T]^T$ $\mathbf{T}_2 = \begin{bmatrix} 2\mathbf{I}_K & \mathbf{0} & \mathbf{I}_K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_K \end{bmatrix}^T$ $\hat{\mathbf{H}} = \mathbf{T}_2\tilde{\mathbf{H}}\mathbf{T}_2^T$ $\hat{\mathbf{p}} = \mathbf{T}_2\tilde{\mathbf{p}}$ $\mathbf{F}_i$: $(4K+1) \times (4K+1)$ diagonal matrix
16-QAM	$\tilde{\mathbf{x}} = \mathbf{T}_2^T\mathbf{y}$ $\mathbf{y} = [\boldsymbol{\alpha}_1^T\ \boldsymbol{\alpha}_2^T]^T$ $\mathbf{T}_2 = [2\mathbf{I}_{2K}\ \mathbf{I}_{2K}]^T$ $\hat{\mathbf{H}} = \mathbf{T}_2\tilde{\mathbf{H}}\mathbf{T}_2^T$ $\hat{\mathbf{p}} = \mathbf{T}_2\tilde{\mathbf{p}}$ $\mathbf{F}_i$: $(4K+1) \times (4K+1)$ diagonal matrix
64-QAM	$\tilde{\mathbf{x}} = \mathbf{T}_3^T\mathbf{y}$ $\mathbf{y} = [\boldsymbol{\alpha}_1^T\ \boldsymbol{\alpha}_2^T\ \boldsymbol{\alpha}_3^T]^T$ $\mathbf{T}_3 = [4\mathbf{I}_{2K}\ 2\mathbf{I}_{2K}\ \mathbf{I}_{2K}]^T$ $\hat{\mathbf{H}} = \mathbf{T}_3\tilde{\mathbf{H}}\mathbf{T}_2^T$ $\hat{\mathbf{p}} = \mathbf{T}_3\tilde{\mathbf{p}}$ $\mathbf{F}_i$: $(6K+1) \times (6K+1)$ diagonal matrix

where $N = \lceil log_2\sqrt{M}\rceil$, i.e., N is equal to the integer immediately greater than or equal to $log_2\sqrt{M}$, $\omega_{n,i}$ are constant coefficients and $\alpha_{n,i} \in \{1, -1\}$ if $\omega_{n,i} \neq 0$ ($n = 1, \ldots, N$; $i = 1, \ldots, 2K$). For instance, for flat rectangular 8-QAM, $\tilde{x}_i = 2\alpha_{1,i} + \alpha_{2,i}$ for $i = 1, \ldots, K$ and $\tilde{x}_i = 0 \cdot \alpha_{1,i} + \alpha_{2,i}$ for $i = K+1, \ldots, 2K$. For square 64-QAM of equal protection, $\tilde{x}_i = 4\alpha_{1,i} + 2\alpha_{2,i} + \alpha_{3,i}$ for $i = 1, \ldots, 2K$.

From (4), it is seen that the detection of $\tilde{x}_i$ is equivalent to the detection of $\alpha_{n,i}$ ($n = 1, \ldots, N$). In the following discussion, $\alpha_{n,i}$ ($n = 1, \ldots, N$) will be referred to as the "index bits" of the corresponding symbol, which is different from the actual information bits that are mapped into the symbol. As such, the detection scheme that is based on the representation in (4) is independent of the constellation mapping from information bits into symbol.

Let

$$\mathbf{y} = [\boldsymbol{\alpha}_1^T\ \boldsymbol{\alpha}_2^T\ \cdots\ \boldsymbol{\alpha}_N^T]^T \tag{5}$$

with $\boldsymbol{\alpha}_n = [\alpha_{n,1}\ \alpha_{n,2}\ \cdots\ \alpha_{n,2K}]^T$ ($n = 1, \ldots, N$). Then

$$\tilde{\mathbf{x}} = \mathbf{T}_N^T\mathbf{y} \tag{6}$$

where

$$\mathbf{T}_N = [\boldsymbol{\Omega}_1\ \boldsymbol{\Omega}_2\ \cdots\ \boldsymbol{\Omega}_N]^T \tag{7}$$

and $\boldsymbol{\Omega}_n = \text{diag}\{\omega_{n,1}, \omega_{n,2}, \ldots, \omega_{n,2K}\}$ ($n = 1, \ldots, N$). Then the problem in (3) can be converted to

$$\text{minimize} \quad \mathbf{y}^T\hat{\mathbf{H}}\mathbf{y} + \mathbf{y}^T\hat{\mathbf{p}} \tag{8a}$$

$$\text{subject to} : y_i \in \{1, -1\} \text{ for } i \in \mathcal{I} \tag{8b}$$

where $\hat{\mathbf{H}} = \mathbf{T}_N\tilde{\mathbf{H}}\mathbf{T}_N^T$, $\hat{\mathbf{p}} = \mathbf{T}_N\tilde{\mathbf{p}}$, and $\mathcal{I} = \text{index}(\boldsymbol{\omega})$ with $\boldsymbol{\omega} = [\boldsymbol{\omega}_1^T\ \boldsymbol{\omega}_2^T\ \cdots\ \boldsymbol{\omega}_N^T]^T$ and $\boldsymbol{\omega}_n = [\omega_{n,1}\ \omega_{n,2}\ \cdots\ \omega_{n,2K}]^T$ ($n = 1, \ldots, N$). The function index($\mathbf{z}$) outputs a set containing all the appropriate indices of $\mathbf{z}$ whose corresponding elements are non-zeros. Therefore, the M-ary detection problem in (3) is converted into a binary detection problem.

Using the SDP relaxation approach adopted in [1]-[4], the binary detection problem in (8) can be approximated by solving the following optimization problem which is given as

$$\text{minimize} \;\; \text{tr}(\mathbf{CY}) \tag{9a}$$
$$\text{subject to: } \mathbf{Y} \succeq \mathbf{0} \tag{9b}$$
$$\text{tr}(\mathbf{F}_i\mathbf{Y}) = 1 \;\; \text{for } i \in \tilde{\mathcal{I}} \tag{9c}$$

where

$$\mathbf{Y} = \begin{bmatrix} \mathbf{y}\mathbf{y}^T & \mathbf{y} \\ \mathbf{y}^T & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \hat{\mathbf{H}} & \hat{\mathbf{p}}/2 \\ \hat{\mathbf{p}}^T/2 & 1 \end{bmatrix} \tag{10}$$

$\tilde{\mathcal{I}} = \mathcal{I} + \{2KN+1\}$, and $\mathbf{F}_i$ is a $(2KN+1) \times (2KN+1)$ diagonal matrix whose diagonal entries are all zeros except the ith entry which is one. The notation $\mathbf{Y} \succeq \mathbf{0}$ denotes that matrix $\mathbf{Y}$ is positive semidefinite. In Table I, $\hat{\mathbf{H}}$, $\hat{\mathbf{p}}$, and $\mathbf{F}_i$ for 8-, 16-, and 64-QAM in the proposed SDP relaxation method are described, where $\mathbf{I}_K$ and $\mathbf{I}_{2K}$ denote identity matrixes of sizes $K \times K$ and $2K \times 2K$, respectively.

Once the solution to the SDP relaxation problem in (9) is obtained, the solution to the problem in (8) can be approximated. There are several different approaches in the literature for the approximation: the direct approach [4], the rank-one approximation approach [1] [3] [4], and the randomization approach [2] [5]. Denote the approximated binary solution to the problem in (8) as $\hat{\mathbf{y}}$. On the basis of $\hat{\mathbf{y}}$, the solution to the problem in (3) can be obtained by using (6), i.e,

$$\hat{\tilde{\mathbf{x}}} = \mathbf{T}_N^T \hat{\mathbf{y}} \tag{11}$$

whose ith ($i = 1, \ \ldots, \ 2K$) component is denoted as $\hat{\tilde{x}}_i$. Thus, based on the shortest Euclidean distance criterion, the ith ($i = 1, \ \ldots, \ K$) component of the M-ary solution to the original problem in (2) is obtained as

$$\hat{b}_i = \Big\{ I_m + jQ_n : \Big| I_m + jQ_n - (\hat{\tilde{x}}_i + j\hat{\tilde{x}}_{i+K}) \Big| \leq \Big| I_{\tilde{m}} + jQ_{\tilde{n}} - (\hat{\tilde{x}}_i + j\hat{\tilde{x}}_{i+K}) \Big|, \forall m, n \text{ and } \tilde{m}, \tilde{n} \Big\}. \tag{12}$$

Efficient interior-point algorithms can be applied to solve the SDP problem in (9) [4], [9] - [11]. A system that solves the SDP problem in (9) to provide a solution to the original detection problem in (2) will be referred to as the M-QAM SDP relaxation (SDPR) detector. The complexity of the M-QAM SDPR detector is $\mathcal{O}\left((2KN+1)^{3.5}\right)$ [11].

IV. Multistage SDP Relaxation Multiuser Detection

Although the M-ary combinatorial problem in (2) can be solved efficiently by SDP relaxation methods as discussed in Section III, the simulation results which will be presented in Section V show that when M is relatively large, e.g. $M \geq 64$, the performance obtained by using the SDP relaxation methods is not near-optimal.

The reason for less satisfactory performance of relatively large number of constellations can be obtained from the transformed detection problem in (8). Note from (4) that coefficient $\omega_{n,i}$ can be treated as the "amplitude" of index bit $\alpha_{n,i}$ in the transformed detection problem in (8). When the constellation is relatively large, the "transmit power" associated with different index bits of the same symbol can differ significantly. For instance, the power ratio between the strongest and the weakest index bits of the same symbol for a constellation of size 256 is 64. This will lead to less reliable detection results for the index bits with smaller coefficients relative to those with larger coefficients of the same symbol in the above proposed detection method.

If the index bits with relatively larger coefficients are detected accurately, then the multiuser interference (MUI) generated by these bits can be reproduced accordingly. This would help to reduce the interference observed by the remaining un-detected index bits and thus improve their detection performance. Otherwise, if the index bits with relatively larger coefficients are detected incorrectly, then the MUI generated by these bits will be represented wrongly, which leads to significant performance degradation in detecting the remaining un-detected index bits. For a reasonably high signal-to-noise ratio (SNR), the error rate in detecting the index bits with larger coefficients is small and this is very likely to help reduce the detection error rates of the later detected index bits with smaller coefficients. This motivation is similar to that of many other decision-aided multiuser detectors [12]. On the basis of above discussion, a multistage approach is developed as follows to improve the performance of the SDPR detector.

Denote $\mathbf{y}_1^*$ as the binary vector solution to the problem in (8) by solving the SDP relaxation problem in (9). Based on $\mathbf{y}_1^*$, we obtain

$$\hat{\boldsymbol{\alpha}}_1 = \mathbf{y}_1^*(1:2K) \tag{13}$$

where $\mathbf{y}_1^*(1:2K)$ denotes the vector formed by the first $2K$ entries in the vector $\mathbf{y}_1^*$. Bringing the values of $\hat{\boldsymbol{\alpha}}_1$ back into the problem in (3), the problem can be modified as

$$\text{minimize} \;\; \tilde{\mathbf{x}}^T\tilde{\mathbf{H}}\tilde{\mathbf{x}} + \tilde{\mathbf{x}}^T\tilde{\mathbf{p}} \tag{14a}$$
$$\text{subject to} : \tilde{x}_i = \omega_{1,i}\hat{\alpha}_{1,i} + \omega_{2,i}\alpha_{2,i} + \cdots + \omega_{N,i}\alpha_{N,i} \tag{14b}$$
$$\alpha_{n,i} \in \{1, \ -1\} \quad \text{if} \;\; \omega_{n,i} \neq 0$$
$$\text{for} \;\; n = 2, \ \ldots, \ N; i = 1, \ \ldots, \ 2K \tag{14c}$$

By using the SDP relaxation method described in Section III, the problem in (14) can be reduced to

$$\text{minimize} \;\; \text{tr}(\mathbf{C}_2\mathbf{Y}_2) \tag{15a}$$
$$\text{subject to: } \mathbf{Y}_2 \succeq \mathbf{0} \tag{15b}$$
$$\text{tr}(\mathbf{F}_{2i}\mathbf{Y}_2) = 1 \;\; \text{for } i \in \tilde{\mathcal{I}}_2 \tag{15c}$$

where

$$\mathbf{C}_2 = \begin{bmatrix} \hat{\mathbf{H}}_2 & \hat{\mathbf{p}}_2/2 \\ \hat{\mathbf{p}}_2^T/2 & 1 \end{bmatrix} \qquad \mathbf{Y}_2 = \begin{bmatrix} \mathbf{y}_2\mathbf{y}_2^T & \mathbf{y}_2 \\ \mathbf{y}_2^T & 1 \end{bmatrix}$$

$\hat{\mathbf{H}}_2 = \mathbf{T}_{N-1}\tilde{\mathbf{H}}\mathbf{T}_{N-1}^T$, $\hat{\mathbf{p}}_2 = \mathbf{T}_{N-1}(\tilde{\mathbf{p}} + 2\tilde{\mathbf{H}} \cdot \boldsymbol{\Omega}_1\hat{\boldsymbol{\alpha}}_1)$, $\mathbf{T}_{N-1} = [\boldsymbol{\Omega}_2 \cdots \boldsymbol{\Omega}_N]^T$, $\mathbf{y}_2 = \left[\boldsymbol{\alpha}_2^T \cdots \boldsymbol{\alpha}_N^T\right]^T$, $\mathbf{F}_{2i}$ is a $(2K(N-1)+1) \times (2K(N-1)+1)$ diagonal matrix whose diagonal entries are all zeros except the ith entry which is one, and $\tilde{\mathcal{I}}_2 = \mathcal{I}_2 + \{2K(N-1)+1\}$ with $\mathcal{I}_2 = \text{index}\left([\boldsymbol{\omega}_2^T \ \cdots \ \boldsymbol{\omega}_N^T]^T\right)$. Denote $\mathbf{y}_2^*$ as the binary vector solution obtained by solving the SDP relaxation problem in (15). Based on $\mathbf{y}_2^*$, we obtain

$$\hat{\boldsymbol{\alpha}}_2 = \mathbf{y}_2^*(1:2K) \tag{16}$$

where $\mathbf{y}_2^*(1:2K)$ denotes the vector formed by the first $2K$ entries in the vector $\mathbf{y}_2^*$.

The above procedure continues in a similar way until the $(N-n+1)$th stage with $n = N,\ (N-1),\ \ldots,\ 1$. Then the problem in (3) can be modified by using the values of $\hat{\alpha}_{1,i},\ \ldots,\ \hat{\alpha}_{N-n,i}$ $(i = 1,\ \ldots,\ 2K)$ obtained from previous stages as follows

$$\text{minimize} \quad \tilde{\mathbf{x}}^T\tilde{\mathbf{H}}\tilde{\mathbf{x}} + \tilde{\mathbf{x}}^T\tilde{\mathbf{p}} \tag{17a}$$

$$\text{subject to}: \tilde{x}_i = \omega_{1,i}\hat{\alpha}_{1,i} + \cdots \omega_{N-n,i}\hat{\alpha}_{N-n,i} \tag{17b}$$

$$+\omega_{N-n+1,i}\alpha_{N-n+1,i} + \cdots + \omega_{N,i}\alpha_{N,i}$$

$$\alpha_{n,i} \in \{1,\ -1\} \quad \text{if} \quad \omega_{n,i} \neq 0 \tag{17c}$$

$$\text{for} \quad n = N-n+1,\ \ldots,\ N; i = 1,\ \ldots,\ 2K$$

and the problem in (17) can be reduced to

$$\text{minimize} \quad \text{tr}(\mathbf{C}_{N-n+1}\mathbf{Y}_{N-n+1}) \tag{18a}$$

$$\text{subject to: } \mathbf{Y}_{N-n+1} \succeq \mathbf{0} \tag{18b}$$

$$\text{tr}(\mathbf{F}_{N-n+1,i}\mathbf{Y}_{N-n+1}) = 1 \quad \text{for } i \in \tilde{\mathcal{I}}_{N-n+1} \tag{18c}$$

where

$$\mathbf{C}_{N-n+1} = \begin{bmatrix} \hat{\mathbf{H}}_{N-n+1} & \hat{\mathbf{p}}_{N-n+1}/2 \\ \hat{\mathbf{p}}_{N-n+1}^T/2 & 1 \end{bmatrix}$$

$$\mathbf{Y}_{N-n+1} = \begin{bmatrix} \mathbf{y}_{N-n+1}\mathbf{y}_{N-n+1}^T & \mathbf{y}_{N-n+1} \\ \mathbf{y}_{N-n+1}^T & 1 \end{bmatrix}$$

$$\hat{\mathbf{H}}_{N-n+1} = \mathbf{T}_n\tilde{\mathbf{H}}\mathbf{T}_n^T$$

$$\hat{\mathbf{p}}_{N-n+1} = \mathbf{T}_n\left[\tilde{\mathbf{p}} + 2\tilde{\mathbf{H}}(\boldsymbol{\Omega}_1\hat{\boldsymbol{\alpha}}_1 + \cdots + \boldsymbol{\Omega}_{N-n}\hat{\boldsymbol{\alpha}}_{N-n})\right]$$

$$\mathbf{T}_n = [\boldsymbol{\Omega}_{N-n+1} \ \cdots \ \boldsymbol{\Omega}_N]^T$$

$$\mathbf{y}_{N-n+1} = \left[\boldsymbol{\alpha}_{N-n+1}^T \cdots \ \boldsymbol{\alpha}_N^T\right]^T$$

$\mathbf{F}_{N-n+1,i}$ is a $(2Kn+1)\times(2Kn+1)$ diagonal matrix whose diagonal entries are all zeros except the ith entry which is one, and $\tilde{\mathcal{I}}_{N-n+1} = \mathcal{I}_{N-n+1} + \{2Kn+1\}$ with $\mathcal{I}_{N-n+1} = \text{index}\left([\boldsymbol{\omega}_{N-n+1}^T \ \cdots \ \boldsymbol{\omega}_N^T]^T\right)$.

Denoting $\mathbf{y}_{N-n+1}^*$ as the binary solution obtained by solving the SDP relaxation problem in (18), then

$$\hat{\boldsymbol{\alpha}}_{N-n+1} = \mathbf{y}_{N-n+1}^*(1:2K) \tag{19}$$

where $\mathbf{y}_{N-n+1}^*(1:2K)$ denotes the vector formed by the first $2K$ entries in the vector $\mathbf{y}_{N-n+1}^*$.

From (5), we have

$$\hat{\mathbf{y}} = [\hat{\boldsymbol{\alpha}}_1^T \ \hat{\boldsymbol{\alpha}}_2^T \ \cdots \ \hat{\boldsymbol{\alpha}}_N^T]^T. \tag{20}$$

As has been mentioned in Section III, the solution to the original problem in (2) can be obtained using (11) and (12).

Therefore, for the M-QAM detection problem, there are N SDP problems of different sizes to solve, which are given in (18). A system that solves the SDP problems in (18) to provide a solution to the original detection problem in (2) will be referred to as the M-QAM multistage SDP relaxation (M-SDPR) detector, which is described in Table II. The complexity involved in the $(N-n+1)$th $(n = N,\ (N-1),\ \ldots,\ 1)$ stage of the M-QAM M-SDPR detector is $\mathcal{O}\left((2Kn+1)^{3.5}\right)$.

TABLE II
MULTISTAGE SDP RELAXATION DETECTOR FOR M-QAM SIGNALS

Step 1: Set $N = \lceil log_2\sqrt{M}\rceil$, $n = N$.
Step 2: Set $\mathbf{T}_n = [\boldsymbol{\Omega}_{N-n+1} \ \cdots \ \boldsymbol{\Omega}_{N-1} \ \ \boldsymbol{\Omega}_N]^T$,
$\hat{\mathbf{p}}_{N-n+1} = \mathbf{T}_n\left[\tilde{\mathbf{p}} + 2\tilde{\mathbf{H}}(\boldsymbol{\Omega}_1\hat{\boldsymbol{\alpha}}_1 + \cdots + \boldsymbol{\Omega}_{N-n}\hat{\boldsymbol{\alpha}}_{N-n})\right]$,
$\mathbf{C}_{N-n+1} = \begin{bmatrix} \mathbf{T}_n\tilde{\mathbf{H}}\mathbf{T}_n^T & \hat{\mathbf{p}}_{N-n+1}/2 \\ \hat{\mathbf{p}}_{N-n+1}^T/2 & 1 \end{bmatrix}$,
$\mathcal{I}_{N-n+1} = \text{index}\left([\boldsymbol{\omega}_{N-n+1}^T \ \cdots \ \boldsymbol{\omega}_N^T]^T\right)$,
and $\tilde{\mathcal{I}}_{N-n+1} = \mathcal{I}_{N-n+1} + \{2Kn+1\}$.
Step 3: Solve the SDP problem
min $\text{tr}(\mathbf{C}_{N-n+1}\mathbf{Y})$
subject to: $\mathbf{Y} \succeq \mathbf{0}$
$y_{ii} = 1 \quad \forall i \in \tilde{\mathcal{I}}_{N-n+1}$
Step 4: Denote the solution to the SDP problem as $\mathbf{Y}^*$.
Based on $\mathbf{Y}^*$, obtain the binary vector solution $\mathbf{y}^*$.
Let $\hat{\boldsymbol{\alpha}}_{N-n+1} = \mathbf{y}^*(1:2K)$.
Step5: $n = n-1$. If $n > 0$, go to Step 2.
Otherwise, go to Step 6.
Step 6: Let $\hat{\mathbf{y}} = [\hat{\boldsymbol{\alpha}}_1^T \ \hat{\boldsymbol{\alpha}}_2^T \ \cdots \ \hat{\boldsymbol{\alpha}}_N^T]^T$, then $\hat{\tilde{\mathbf{x}}} = \mathbf{T}_N^T\hat{\mathbf{y}}$.
Step 7: According to (12) obtain the solution to the original problem in (2).

V. SIMULATION RESULTS

Computer simulations were conducted to evaluate the performance of the proposed SDP relaxation detectors in terms of SER. The MATLAB LMI control toolbox was used in the simulations to implement the SDPR and the M-SDPR detectors [13].

The uplink of a multicarrier code-division multiple-access (MC-CDMA) multiuser communication system with two-ray frequency-selective multipath Rayleigh fading channel is considered [14]. The attenuation magnitudes for Rayleigh fading are changed once over each symbol period. All users in the simulated systems are assumed to have equal average transmission signal energy. The spreading codes of all users are of length fifteen and are randomly generated by following a Gaussian distribution with zero mean and unit variance. The same spreading code is shared by both I and Q components for each user.

In the first example we considered a four-user system using 16-QAM. Our simulations show that among the three approximation approaches presented in Section III, the randomization approach often provides better SER performance than the direct and the rank-one approaches in approximating the solution to the original problem based on the solution to the SDP relaxation problem. This result is consistent with those in [5] and [6]. Therefore, without otherwise specified, we will use the randomization approach to obtain the final results of SDP relaxation detectors in the following simulations, and the results shown are averages of those obtained from 40 independent randomizations.

In Fig. 1, the SERs obtained by using the SDPR and the M-SDPR detectors are plotted. For comparison purposes, the SERs obtained by using the optimum ML detector [8], the Equal Gain Combining (EGC) detector, the Maximal Ratio Combining (MRC) detector [14], the Global Minimum Mean-Square-Error (GMMSE) detector [15], and the SDP relaxation detector proposed in [6] are also plotted in the same figure. It can be observed in Fig. 1 that the detection performance of the SDPR and the M-SDPR detectors is consistently very close to each other and to that of the ML detector and the detector proposed in [6], and is superior relative to that of the GMMSE detector. In addition, the M-SDPR detector delivers slightly lower SER than the SDPR detector.

In the second example we considered a four-user system

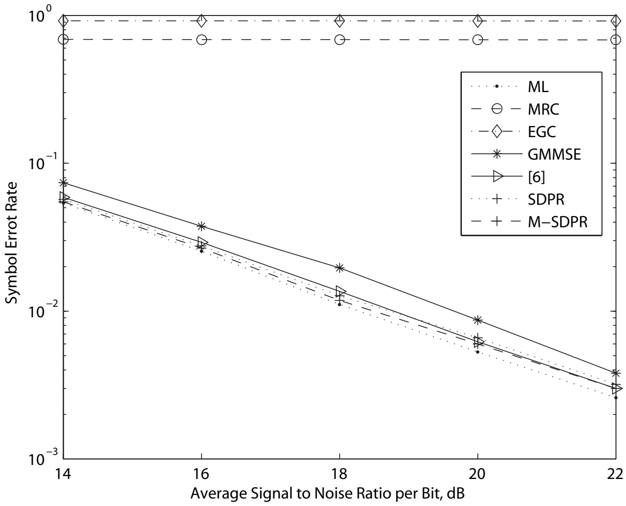

Fig. 1. SERs of the 16-QAM system (example one).

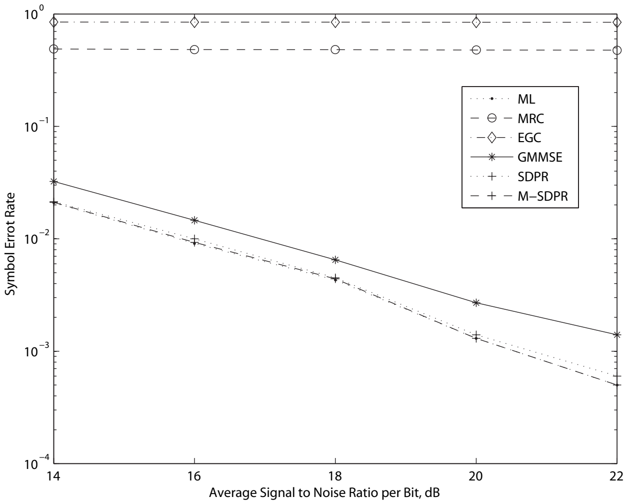

Fig. 2. SERs of the 8-QAM system (example two).

using 8-QAM as modulation scheme. The SERs obtained by using the ML, the SDPR, the M-SDPR, the MRC, the EGC, and the GMMSE detectors are plotted in Fig. 2. As similar to what has been observed in the first example, the detection performance of the SDPR and the M-SDPR detectors is superior relative to that of the GMMSE detector.

In the third example we considered a four-user system using 64-QAM as modulation scheme, and the SERs obtained are plotted in Fig. 3. In this example, the SER of the M-SDPR detector was obtained by using the rank-one approximation approach, which in this case provides the best SER performance among the three approximation approaches presented in Section III. It can be observed in Fig. 3 that the detection performance of the SDPR detector is worse than that of the GMMSE detector and the SDPR detector can not work properly in this system scenario. However, the proposed M-SDPR detector maintains good performance, which is superior relative to that of the GMMSE detector.

VI. Conclusions

New multiuser detectors based on SDP relaxation approach have been proposed for systems with M-QAM. In the proposed

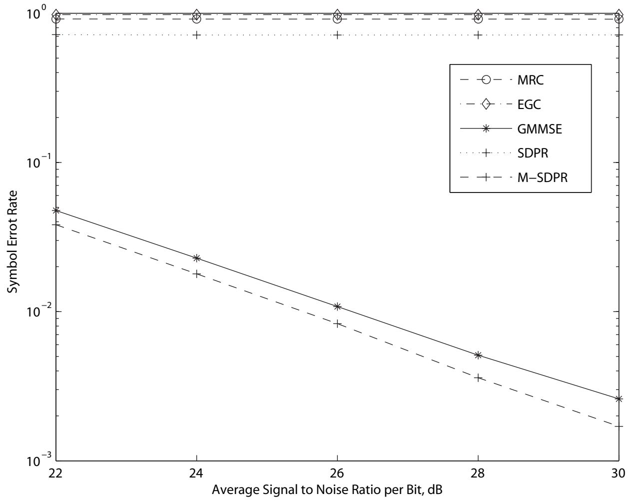

Fig. 3. SERs of the 64-QAM system (example three).

approach, the combinatorial problem associated with the M-QAM ML detection is relaxed to an SDP problem, which leads to detectors of polynomial complexity. A multistage approach has also been proposed to improve the performance of the proposed SDP relaxation based detectors. Computer simulations have been presented to demonstrate that the proposed multistage SDP relaxation detectors offer SER performance which outperforms that of several existing suboptimal detectors.

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