LETTER

Sparse Recovery Algorithms for Pilot Assisted MIMO OFDM Channel Estimation

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SUMMARY In this letter, the sparse recovery algorithm orthogonal matching pursuit (OMP) and subspace pursuit (SP) are applied for MIMO OFDM channel estimation. A new algorithm named SOMP is proposed, which combines the advantage of OMP and SP. Simulation results based on 3GPP spatial channel model (SCM) demonstrate that SOMP performs better than OMP and SP in terms of normalized mean square error (NMSE).

key words: *MIMO, OFDM, channel estimation*

1. Introduction

In traditional least square (LS) OFDM channel estimation, we must acquire channel frequency response at pilot positions and then use these observations to interpolate the rest of the subcarriers. Generally, accurate channel estimation requires more pilots than unknown channel coefficients. When channel has large delay spread and contains abundant multipaths, the pilot number raises rapidly. For MIMO, the overhead of pilot symbols becomes considerable as transmitting antennas increase. Therefore, one possible solution is to assume the channel sparsity as *a priori*. Wireless channels in practice are typically sparse. Channel impulse response usually presents to be a large number of taps with very few of them nonzero. With sparse recovery algorithms, the number of pilots can be substantially reduced. Some published work has already shown progress in this field [1], [2]. Matching pursuit (MP) [3] and orthogonal matching pursuit (OMP) [4] are commonly employed, which sequentially identifies a small subset of nonzero taps. Although the algorithms are suboptimal and greedy in nature, they are efficient in terms of performance and complexity. They have been proved to be more accurate than LS approach with less pilots.

In this letter, we formulate OFDM frequency domain channel estimation as a sparse recovery problem and apply MP, OMP, and subspace pursuit (SP) algorithms. After that, we propose a subspace orthogonal matching pursuit (SMOP) algorithm combining the advantage of OMP and SP. Random pilot placement is adopted according to restricted isometry property (RIP) [5]. With frequency orthogonal pilot placement, we extend our work to MIMO OFDM. 3GPP spatial channel model (SCM) [6] is applied in our simulations.

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2. System Model

We consider a multipath environment with S clusters or scatters. The channel impulse response between the i-th transmitter and the j-th receiver is modeled as

$$h_{ji}(\tau, t) = \sum_{p=1}^{S} \alpha_p^{ji}(t)\delta(\tau - \tau_p(t)) \tag{1}$$

where $\alpha_p^{ji}(t) \in \mathbb{C}$ and $\tau_p(t) \in \mathbb{R}^+$ are complex-valued magnitude and real-valued delay spread for path p, respectively. With block-fading channel assumption where the channel parameters are constant over each block and assuming perfect symbol synchronization, the equivalent discrete impulse response of the channel can be modeled as

$$h_{ji}(m) = \sum_{p=1}^{S} \alpha_p^{ji}\delta((m - \tau_p)T_s) \tag{2}$$

where T_s is the sampling interval of the system. We notice that in high data rate communication systems where T_s is very small compared to the maximum delay spread, (2) results in a channel with a relatively few nonzero taps. Assuming total channel taps to be L and S of them nonzero ($S \ll L$), we call it S-sparse channel.

Considering an OFDM system with N subcarriers, among which N_p subcarriers are selected as pilots with positions represented by $k_1, k_2, \ldots, k_{N_p}$ ($1 \le k_1 < k_2 < \ldots < k_{N_p} \le N$) and N_d ($N_d = N - N_p$) subcarriers are used for data transfer. We denote the transmitted pilot symbols and the received pilot symbols as $X(k_1), X(k_2), \ldots, X(k_{N_p})$ and $Y(k_1), Y(k_2), \ldots, Y(k_{N_p})$, respectively. The estimated transfer function on pilot subcarriers is

$$\hat{H}(m) = \frac{Y(m)}{X(m)}, \quad m = k_1, k_2, \ldots, k_{N_p} \tag{3}$$

Then we make linear interpolation or cubic spline interpolation between each two neighboring pilot subcarriers and get the channel transfer function $\hat{H}(m)$ ($m = 1, 2, \ldots, N$), which approximates Discrete Fourier Transform (DFT) of the channel impulse response as defined in (2). In order to make use of channel sparsity, we formulate the problem as

$$\begin{bmatrix} Y(k_1) \\ Y(k_2) \\ \vdots \\ Y(k_{N_p}) \end{bmatrix} = \begin{bmatrix} X(k_1) & 0 & 0 & 0 \\ 0 & X(k_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & X(k_{N_p}) \end{bmatrix} \cdot \boldsymbol{F}_{N_p \times L} \cdot \begin{bmatrix} h(1) \\ h(2) \\ \vdots \\ h(L) \end{bmatrix} + \begin{bmatrix} n(1) \\ n(2) \\ \vdots \\ n(N_p) \end{bmatrix} \quad (4)$$

where $[h(1), h(2), \cdots, h(L)]^{\mathrm{T}}$ is the channel impulse response, $\left[n(1), n(2), \cdots, n(N_p)\right]^{\mathrm{T}}$ is the noise vector with each element to be an AWGN variable and

$$\boldsymbol{F}_{N_p \times L} = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & \omega^{k_1} & \cdots & \omega^{k_1 \cdot (L-1)} \\ 1 & \omega^{k_2} & \cdots & \omega^{k_2 \cdot (L-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{k_{N_p}} & \cdots & \omega^{k_{N_p} \cdot (L-1)} \end{bmatrix}$$

where $\omega = e^{-j2\pi/N}$. Actually $\boldsymbol{F}_{N_p \times L}$ is a submatrix selected by row index $\left[k_1, k_2, \cdots, k_{N_p}\right]$ and column index $[0, 1, \cdots, L-1]$ from a standard $N \times N$ Fourier matrix. For simplicity, we denote $\boldsymbol{n} = \left[n(1), n(2), \cdots, n(N_p)\right]^{\mathrm{T}}$, $\boldsymbol{h} = [h(1), h(2), \cdots, h(L)]^{\mathrm{T}}$, $\boldsymbol{y} = \left[Y(k_1), Y(k_2), \cdots, Y(k_{N_p})\right]^{\mathrm{T}}$, $\boldsymbol{X} = diag\left\{X(k_1), X(k_2), \cdots, X(k_{N_p})\right\}$ and $\boldsymbol{A} = \boldsymbol{X} \cdot \boldsymbol{F}_{N_p \times L}$, so

$$\boldsymbol{y} = \boldsymbol{A} \cdot \boldsymbol{h} + \boldsymbol{n} \quad (5)$$

It's observed from Eq. (5) that the purpose of channel estimation is to obtain $\boldsymbol{h}$ from $\boldsymbol{y}$ and $\boldsymbol{A}$. If rows of $\boldsymbol{A}$ is more than its columns ($N_p > L$), Eq. (5) is a standard LS problem with its solution

$$\hat{\boldsymbol{h}}_{LS} = (\boldsymbol{A}^H \boldsymbol{A})^{-1} \boldsymbol{A}^H \cdot \boldsymbol{y} \quad (6)$$

Obviously, we are more interested in the case when the pilots are less than the channel coefficients ($N_p < L$). It's significantly appealing in reducing pilots and thus improving the spectral efficiency. Theoretically, there's feasible solution for sparse recovery problem [5] if most elements of vector $\boldsymbol{h}$ are zero ($S \ll L$).

Since MIMO OFDM channel estimation can be decomposed into simultaneously estimating of several SISO OFDM channels where we employ frequency orthogonal pilot placement for different transmitters, we will mainly focus on sparse recovery algorithms for each SISO OFDM channel in the following section.

3. Sparse Recovery Algorithms

A collection of sparse recovery algorithms has recently emerged with the name compressive sensing [7], which enables efficient reconstruction of sparse signals from relatively few linear measurements. A S-sparse vector $\boldsymbol{h} \in \mathbb{R}^L$ can be recovered from Eq. (5) with deliberately designed $\boldsymbol{A} \in \mathbb{R}^{N_p \times L}$ by solving ℓ_0-norm minimization problem

$$\min_{\boldsymbol{h} \in \mathbb{R}^L} \|\boldsymbol{h}\|_0 \;\text{ s.t. }\; \|\boldsymbol{y} - \boldsymbol{A} \cdot \boldsymbol{h}\|_2 \le \sigma \quad (7)$$

where $\|\boldsymbol{h}\|_0$ counts the number of nonzero components of $\boldsymbol{h}$ and $S \le N_p \le L$. This problem is combinatorial and NP hard. However, Candes, Tao, Donoho, Tropp and their colleagues have shown that it can be replaced by a convex optimization problem [7], [8]

$$\min_{\boldsymbol{h} \in \mathbb{R}^L} \|\boldsymbol{h}\|_1 \;\text{ s.t. }\; \|\boldsymbol{y} - \boldsymbol{A} \cdot \boldsymbol{h}\|_2 \le \sigma \quad (8)$$

where σ is the variance of noise $\boldsymbol{n}$. Methods for solving above and closely related problems can be roughly divided into two classes, including greedy algorithms and convex optimization algorithms. Here we mainly focus on greedy algorithms due to its low complexity. In application of time-varying channel where channel estimation is frequently carried out, it's inappropriate to choose high computational convex optimization algorithms.

3.1 MP and OMP

MP [3] is a sort of algorithm that constructs a linear combination of matrix columns closest to the signal. At each step, a new column that best correlates with the current residue is added to current selection. Then it updates the residue by projecting it onto the new selection. Although MP can rapidly find an approximation with asymptotic convergence, its shortcoming lies in the fact that it may select the same columns several times which lowers the efficiency. Hence, OMP [4] has been proposed as a revised MP by only using residue's orthogonal component for next iteration. Only the component that is orthogonal with the space spanned by the previous selected columns is preserved. OMP has proved to be one of the most efficient algorithms for sparse recovery problem [9]. In practical applications, OMP is always the best candidate because of its reasonable tradeoff between performance and complexity. The shortcoming of OMP lies in its unidirectional adding new columns without removing out-dated columns. When a selection error occurs, the iteration will continue to the end without correcting them adaptively. Aiming at these disadvantages, SP have been proposed.

3.2 SP

The idea of SP is to iteratively refine S columns selection from the dictionary matrix through LS method until the stop condition is satisfied [10]. Although it still relies on greedy rule for column selection, it allows new columns to enter into as well as leave the selection set. At each step, it selects S columns rather than only one column as in MP and OMP. The subspace spanned by S columns is thus tracked down, which is the implication of its name. The weak point of SP is that we should know S before the start of the algorithm. Otherwise we can only guess a value for S. So it's necessary to extend SP to the occasion where the sparsity is unknown.

3.3 SOMP

The stop condition for OMP employs the threshold that equals to the noise variance, while the counterpart for SP only relies on previous iterative result. Apparently the latter is more appealing since it can iteratively refine the result. Besides, SP allows the columns to enter into as well as leave the selection set, which is the chief drawback for OMP. At each iteration, OMP always greedily selects one column vector, while SP selects several columns in batch. The possibility to correctly find one column with one selection is much lower than with batch selection. As a result, we combine the advantage of OMP and SP and propose SOMP. The intention is first to identify the sparsity using OMP and then to find a sparse solution with SP. By the way the recovery result of OMP indicating the column selection can also be adopted for the first iteration of SP.

Definition: If matrix $\boldsymbol{M}$ satisfies that $\boldsymbol{M}^H\boldsymbol{M}$ is invertible, we define the orthogonal part of $\boldsymbol{y}$ on $\boldsymbol{M}$ to be

$$\mathrm{orthg}(\boldsymbol{y}, \boldsymbol{M}) \triangleq \boldsymbol{y} - \boldsymbol{M}\boldsymbol{M}^{\dagger}\boldsymbol{y} \tag{9}$$

where

$$\boldsymbol{M}^{\dagger} = (\boldsymbol{M}^H\boldsymbol{M})^{-1}\boldsymbol{M}^H \tag{10}$$

is the pseudo inverse of $\boldsymbol{M}$.

Algorithm 1 SOMP

Input: $\boldsymbol{A}, \boldsymbol{y}, \sigma$

1. Normalize columns of $\boldsymbol{A}$:
   Normalize each column of matrix $\boldsymbol{A}$ with a coefficient diagonal matrix $\boldsymbol{C}$ so that $\boldsymbol{A} = \boldsymbol{D}\cdot\boldsymbol{C}$
2. Identify the sparsity:
   Initialization:
   $\boldsymbol{r}_1 = \boldsymbol{y}, I_1 = \varnothing, I_1^c = \{1, \ldots, L\}$
   Iteration: $k = 1, 2, \ldots$
   $m_k = \arg\max_{i\in I_k^c} | < \boldsymbol{D}(i), \boldsymbol{r}_k > |$
   $\boldsymbol{u}_k = \boldsymbol{D}(m_k) - \sum_{i\in I_k} < \boldsymbol{D}(m_k), \boldsymbol{u}_i > \frac{\boldsymbol{u}_i}{\|\boldsymbol{u}_i\|_2^2}$
   $\boldsymbol{r}_k = < \boldsymbol{r}_k, \boldsymbol{u}_k > \frac{\boldsymbol{u}_k}{\|\boldsymbol{u}_k\|_2^2} + \boldsymbol{r}_{k+1}$
   if $\|\boldsymbol{r}_{k+1}\|_2 \le \sigma$, break
   $I_{k+1} = \{I_k, m_k\}, I_{k+1}^c = I_k^c\backslash\{m_k\}$
   Store I_{k+1} in $\hat{I}$
3. Get the sparse solution:
   Initialization:
   $S = \|\hat{I}\|_0, \boldsymbol{d}_1 = \mathrm{orthg}(\boldsymbol{y}, \boldsymbol{D}(\hat{I}))$
   Iteration: $k = 1, 2, \ldots$
   If $\boldsymbol{d}_k = 0$, break
   $\boldsymbol{z} = \boldsymbol{D}^H\boldsymbol{d}_k$
   $I_p = \left\{(l_1, \ldots, l_S) : |\boldsymbol{z}(l_1)| \ge \cdots \ge |\boldsymbol{z}(l_S)| \ge \cdots \ge |\boldsymbol{z}(l_L)|\right\}$
   $I' = \hat{I} \bigcup I_p, \boldsymbol{w} = \boldsymbol{D}^{\dagger}(I')\boldsymbol{y}$
   $I_q = \left\{(l_1, \ldots, l_S) : |\boldsymbol{w}(l_1)| \ge \cdots \ge |\boldsymbol{w}(l_S)| \ge \cdots \ge |\boldsymbol{w}(l_L)|\right\}$
   $\boldsymbol{d}_{k+1} = \mathrm{orthg}(\boldsymbol{y}, \boldsymbol{D}(I_q))$
   If $\|\boldsymbol{d}_{k+1}\|_2 > \|\boldsymbol{d}_k\|_2$, break
   $\hat{I} = I_q$
   Store $\boldsymbol{x}$: $\boldsymbol{x}(\hat{I}) = \boldsymbol{D}^{\dagger}(\hat{I})\boldsymbol{y}, \quad \boldsymbol{x}(\hat{I}^c) = \boldsymbol{0}$
4. Output:
   $\hat{\boldsymbol{h}}_{somp} = \boldsymbol{C}^{-1}\boldsymbol{x}$

We describe the SOMP algorithm as follows. First we normalize each columns of $\boldsymbol{A}$ and get a coefficient diagonal matrix $\boldsymbol{C}$. Then we start to identify the sparsity of the solution. Index set I_1 indicating current selected columns is initialized to be empty while its complementary set I_1^c is $\{1, 2, \ldots, L\}$. Current residue $\boldsymbol{r}_1$ is initialize to be $\boldsymbol{y}$. At k-th iterative step, we select a column index m_k from I_k^c so that $\boldsymbol{D}(m_k)$ has the largest inner product with current residue $\boldsymbol{r}_k$. Gram-Schmidt orthogonalization is implemented on $\boldsymbol{D}(m_k)$ to remove the component inside the column space spanned by I_k. $\{\boldsymbol{u}_k\}$ is an iteratively generated set which can be regarded as unnormalized base vectors for the space spanned by I_k. Then we update $\boldsymbol{r}_k$ by projecting it on this space. If the stop condition $\|\boldsymbol{r}_{k+1}\|_2 \le \sigma$ is satisfied, we break the iteration and store I_{k+1} in $\hat{I}$. Otherwise we update I_{k+1} by adding m_k into I_k; meanwhile its complementary set I_{k+1}^c is also updated. After that, we enter into the stage for sparse solution. The identified sparsity S is initialized to be the size of $\hat{I}$. $\boldsymbol{d}_1$ is initialized as the orthogonal part of $\boldsymbol{y}$ on $\boldsymbol{D}(\hat{I})$, where $\boldsymbol{D}(\hat{I})$ is defined as the submatrix from $\boldsymbol{D}$ with its columns indexed by $\hat{I}$. At k-th iterative step, we first check whether $\boldsymbol{d}_k$ is zero. If so, we break the iteration. Otherwise, we project $\boldsymbol{d}_k$ onto $\boldsymbol{D}$, from which we pick up S largest components and store their indices in I_p. The union of $\hat{I}$ and I_p is denoted as I'. From $\boldsymbol{D}^{\dagger}(I')\boldsymbol{y}$ we picking up S largest components and refine I_q. Let $\boldsymbol{d}_{k+1}$ denote the orthogonal part of $\boldsymbol{y}$ on $\boldsymbol{D}(I_q)$. If $\|\boldsymbol{d}_{k+1}\|_2$ appears to be greater than the last step, it means the orthogonal part can't be smaller. We break the iteration. Otherwise we update $\hat{I}$ by I_q. When out of iteration, $\boldsymbol{x}$ is yielded with nonzero components indexed by $\hat{I}$ satisfying $\boldsymbol{x}(\hat{I}) = \boldsymbol{D}^{\dagger}(\hat{I})\boldsymbol{y}$ and $\boldsymbol{C}^{-1}\boldsymbol{x}$ is output as the final solution.

In practice, channel multipath number is usually unknown. It do exists that OMP gets a wrong sparse solution while its identification on sparsity is correct. In this case SOMP continues with batch selection and eventually gets a correct recovery. In optimization research field [11], [12], it's common to use two steps for sparse recovery, with the first to identify the sparsity and the second to decide corresponding value. The difference is that SOMP uses a combined greedy algorithm instead of high computational optimization algorithm. Therefore, SOMP may offer one point of view to develop similar low complexity sparse recovery algorithms.

4. Simulation Results

In our simulations, we consider MIMO OFDM system with two transmitting antennas and two receiving antennas. The channels are generated using 3GPP SCM [6], where we apply MP, OMP, SP and SOMP algorithms for channel estimation. According to [5], we can randomly place pilots among all OFDM subcarriers so that the matrix $\boldsymbol{A}$ in (5) satisfies RIP. Meanwhile, with the purpose to simplify MIMO OFDM channel estimation problem, we employ frequency orthogonal pilot placement for different transmitting antennas. For example, we place 12 pilots among 256 subcarriers for two different transmitting antennas. First we randomly

Table 1 System parameters.

Number of transmitting antennas	$N_t = 2$
Number of receiving antennas	$N_r = 2$
Number of total subcarriers	$N = 256$
Number of pilot subcarriers	$N_p = 12$
Number of cyclic prefix	$N_G = 64$
Number of channel multipaths	$S = 5$
Length of channel impulse response	$L = 40$
Modulation	QPSK

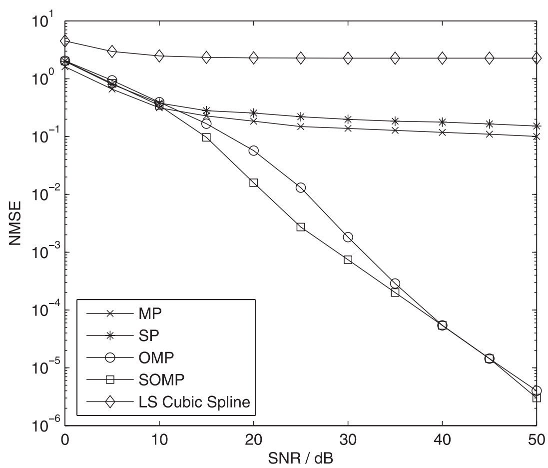

Fig. 1 NMSE vs. SNR with unknown sparsity.

select $2 \times 12 = 24$ positions from 256 positions. Then we randomly choose 12 positions from the selected 24-position set for one transmitter. The remaining 12 positions are used for another transmitter's pilot placement. Thus pilots for different antennas are un-overlapped and do not interfere each other. 2×2 MIMO OFDM channel estimation problem is decoupled into 4 SISO OFDM channel estimation problems. Hence we can concentrate on sparse channel estimation for each SISO OFDM.

System parameters in our simulations are listed in Table 1. Pilot positions used in simulations are [6, 20, 36, 58, 70, 90, 118, 169, 182, 202, 223, 240]. Performance comparisons of MP, OMP, SP, SOMP and LS with cubic spline interpolation in terms of normalized mean square error (NMSE) are illustrated in Fig. 1. We define NMSE as

$$\mathrm{NMSE}\{\hat{\boldsymbol{h}}\} = \frac{\|\hat{\boldsymbol{h}} - \boldsymbol{h}\|_2^2}{\|\boldsymbol{h}\|_2^2}$$

where $\hat{\boldsymbol{h}}$ is the estimation of $\boldsymbol{h}$.

It's observed from Fig. 1 that LS cubic spline which does not take channel sparsity as *a prori* performs much worse than sparse recovery algorithms. SOMP has the best performance especially for SNR from 15 dB to 30 dB. In this range, the noise is unnegligible and it disturbs the correct selection for OMP. The reason is the one selection manner and greedy essence of OMP as we analyzed in Sect. 3.3. When SNR is beyond 35 dB, the noise is much smaller compared to the signal component, in this case OMP can also do the right thing as SOMP. For SNR lower than 15 dB, OMP and SOMP are difficult to distinguish because the noise deteriorates both algorithms. Consequently, SOMP combines the advantages of OMP to identify the sparsity and SP to refine the best group selection. It has been demonstrated to be an appropriate candidate for low complexity sparse recovery.

5. Conclusions

This letter studied the sparse recovery algorithms for pilot assisted MIMO OFDM channel estimation, where MP, OMP and SP are applied and SOMP is proposed. Further work will continue on sparse recovery algorithms and complexity reduction will be emphasized.

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