Asymptotic Eigenvalue Density for the Quotient Ensemble of Wishart Matrices

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***Abstract*— The matrix model involving the quotient of two Wishart matrices appears naturally in the investigation of multiple-input multiple-output (MIMO) multiple-access and relay channels. The available exact result for the eigenvalue density for this matrix model involves the determinant of a matrix whose dimension is related to the number of antennas in the channel. Consequently, the exact result becomes impractical while dealing with the case of large number of antennas. In this letter, we derive a novel expression for the asymptotic eigenvalue density of the quotient matrix model. This result is analogous to the Marčenko–Pastur density and can be conveniently applied to deal with the case of large matrix dimensions. Remarkably, satisfactorily results are obtained even for small matrix dimensions. As an application of our asymptotic result, we obtain an upper bound for the ergodic capacity in a full-duplex multi-hop decode-and-forward MIMO relay network.**

***Index Terms*— MIMO, full-duplex, ergodic capacity, Wishart matrix, quotient ensemble.**

I. Introduction

IT IS now well acknowledged that techniques involving a very large number of antennas, such as massive MIMO, will play a key part in the communication networks of the future. Therefore, it is of utmost importance to seek asymptotic results associated with the channel matrix appearing in the communication model, e.g., asymptotic eigenvalue density expressions similar to the celebrated Marčhenko-Pastur formula [1].

In the investigation of multiple-input multiple-output (MIMO) multiple-access and relay channels, a matrix model involving the quotient of two Wishart matrices [2]–[4] arises naturally [5], [6]. The corresponding exact results for the eigenvalue densities exhibit rich mathematical structure associated with biorthogonal ensembles of random matrices [7]. These involve determinants of matrices whose dimensions relate to the number of antennas in the present context. These results are quite useful in evaluating various performance metrics when a moderate number of antennas are involved. However, they become impractical when dealing with a large number of antennas.

In this work, we identify the quotient of two Wishart matrices to belong to the class of *algebraic random matrices*, as defined by Rao and Edelman [8]. The Stieltjes transform of the limiting eigenvalue density for these random matrices is algebraic, i.e., it satisfies a polynomial equation. Therefore, by employing the polynomial method developed in [8], we derive a novel closed-form asymptotic result for the probability density function (PDF) of a generic eigenvalue for the quotient ensemble. The asymptotic density, while expected to work in the large matrix dimension limit, provides good results even for small dimensions. The most difficult part in deriving this exact and closed-form result is identifying the "correct" root of the cubic equation (with quite lengthy and complicated coefficients) satisfied by the Stieltjes transform associated with the PDF, and then applying the Stieltjes-Perron inversion formula. This nontrivial aspect of isolating the correct solution and thereby arriving at a closed-form result has been emphasized by Rao and Edelman [8]. In fact, in many cases one has to rely on numerical approaches to obtain the PDF, or to enumerate only the moments in closed forms.

As an application of our result, we obtain an upper bound for the ergodic capacity in a multi-hop decode-and-forward MIMO full-duplex relay network, with the assumption that each hop and the self-interference channels are subjected to Rayleigh fading. This results in a matrix model comprising the quotient of two Wishart matrices [5], [6], [9]. To evaluate various performance metrics in this context, one requires the eigenvalue statistics for this matrix model. The corresponding exact results have been derived in [6] and [7], and used in [6] to compute the ergodic capacity and outage probability for a two-user MIMO multiple access channel (MAC), considering the individual channels subject to Rayleigh fading.

Our analytical result can also be used in other problems such as the derivation of the ergodic capacity in Non-Orthogonal Multiple Access (NOMA) systems [10] or the performance of the minimum mean square error (MMSE) receiver.

II. Asymptotic Density of the Quotient Ensemble

Consider two $n \times n$ complex Wishart matrices $\mathbf{W}_A$ and $\mathbf{W}_B$ from the distributions $\mathcal{CW}_n(n_A, \mathbf{I}_n)$ and $\mathcal{CW}_n(n_B, \mathbf{I}_n)$, respectively. These matrices have degrees of freedom n_A and n_B, respectively, and the identity matrix $\mathbf{I}_n$ as the covariance matrix. The quotient ensemble of Wishart matrices is defined using the matrix model [7],

$$\mathbf{Q} = (a/n_A)\mathbf{W}_A \left[\mathbf{I}_n + (b/n_B)\mathbf{W}_B\right]^{-1}, \tag{1}$$

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TABLE I
SUCCESSIVE BIVARIATE POLYNOMIALS

Matrix	**Bivariate polynomial**
$\mathbf{A}_1 = \mathbf{I}_n$	$L_1(s,z) = (1-z)s - 1$
$\mathbf{A}_2 = (b/n_B)\mathbf{W}_B \cdot \mathbf{A}_1$	$L_2(s,z) = L_1\Big(bs(1-\beta-\beta zs), \frac{z}{b(1-\beta-\beta zs)}\Big)$
$\mathbf{A}_3 = \mathbf{I}_n + \mathbf{A}_2$	$L_3(s,z) = L_2(s, z-1)$
$\mathbf{A}_4 = \mathbf{A}_3^{-1}$	$L_4(s,z) = L_3\Big(-z - z^2 s, \frac{1}{z}\Big)$
$\mathbf{A}_5 = (a/n_A)\mathbf{W}_A \cdot \mathbf{A}_4$	$L_5(s,z) = L_4\Big(as(1-\alpha-\alpha zs), \frac{z}{a(1-\alpha-\alpha zs)}\Big)$

where a and b are some non-negative scalars. We highlight that (1) is different from the matrix models $\mathbf{W}_A(\mathbf{W}_A + \mathbf{W}_B)^{-1}$ and $\mathbf{W}_A\mathbf{W}_B^{-1}$ which lead to the matrix variate beta distributions of the first kind and the second kind, respectively [11]–[13].

The exact PDF of a generic eigenvalue λ of $\mathbf{Q}$ is given by [7]

$$p(\lambda) = Ce^{-n_A\lambda/a}\lambda^{n_A-n}\det\begin{bmatrix} 0 & [\lambda^{k-1}]_{k=1}^{n} \\ [f_j(\lambda)]_{j=1}^{n} & [h_{j,k}]_{j,k=1}^{n}\end{bmatrix}, \quad (2)$$

where $1/C = -n\,(a/n_A)^{\omega_1}\,(b/n_B)^{\omega_2}\prod_{j=1}^{n}\Gamma(j)\Gamma(n_A - j + 1)$, with $\omega_1 = nn_A - n(n-1)/2$, $\omega_2 = nn_B$, $f_j(\lambda) = U\left(n_B - j + 1, n_A + n_B - j + 2; n_A\lambda/a + n_B/b\right)$, and $h_{j,k} = \Gamma(n_A - n + k)(a/n_A)^{n_A-n+k} \times U\left(n_B - j + 1, n_B + n - j - k + 2; n_B/b\right)$, with $U(\cdot,\cdot;\cdot)$ being the confluent hypergeometric function of the second kind. We emphasize that expression (2) involves a determinant and, hence, is not well suited for large n evaluations. Therefore, we aim to find an asymptotic result for the eigenvalue density.

We consider the scenario $n, n_A, n_B \to \infty$, such that $n/n_A = \alpha \leq 1$ and $n/n_B = \beta \leq 1$. The matrix model (1) falls under the class of algebraic random matrices, as defined in [8], and therefore we can adopt the polynomial method developed therein. The key quantity for this calculation is the Stieltjes transform associated with an asymptotic eigenvalue density $\widetilde{p}(\lambda)$,

$$s(z) = \int_0^\infty \frac{\widetilde{p}(\lambda)}{\lambda - z}\,d\lambda, \quad (3)$$

$z \in \mathbb{C} \setminus \mathbb{R}$, with the corresponding Stieltjes-Perron inversion formula [14],

$$\widetilde{p}(\lambda) = \frac{1}{\pi}\lim_{\epsilon\to 0^+}\Im\left[s(\lambda + i\epsilon)\right]. \quad (4)$$

To obtain the Stieltjes transform $s(z)$ of the asymptotic eigenvalue density for $\mathbf{Q}$, we follow the methodology laid out in [8]. It involves constructing the composite matrix model from its constituents and calculating the associated bivariate polynomials $L(s,z)$. By solving $L(s,z) = 0$ for s, in terms of z, one obtains the Stieltjes transform $s(z)$ for the desired matrix model. In Table I, we compile the implemented sequence of operations. From the table, we see that $\mathbf{Q} = \mathbf{A}_5$ and, therefore, the Stieltjes transform for the associated eigenvalue density is obtained by solving $L_5(s,z) = 0$.

After some rearrangement of terms and simplification, we obtain the following cubic equation to solve,

$$c_3 z^2 s^3 + c_2 z s^2 + c_1 s + c_0 = 0. \quad (5)$$

Here, the z-dependent coefficients are $c_0(z) = b\beta z - a(1+b)(1-\alpha)$, $c_1(z) = 2\,b\beta z^2 + a(\alpha b\beta + 2\alpha b - b\beta + 2\alpha - b - 1)z + a^2(1-\alpha)^2$, $c_2(z) = b\beta z^2 + a(2\alpha b\beta + \alpha b - b\beta + \alpha)z - 2a^2\alpha(1-\alpha)$, and $c_3(z) = a\alpha(b\beta z + a\alpha)$.

Once $s(z)$ is determined, we may apply (4) to obtain $\widetilde{p}(\lambda)$. However, this requires the nontrivial task of identifying the "correct" $s(z)$ out of the three roots of (5). Fortunately, for a cubic equation, the roots are expressible in terms of radicals, and Cardano's method can be used to arrive at the explicit results. After some effort we are able to identify the correct $s(z)$ and then carefully perform the Stieltjes-Perron inversion. We adopt one of the standard approaches to write down the solution of a cubic equation [15], and define

$$G(\lambda) = \left[\Big(\eta(\lambda) + \sqrt{\eta^2(\lambda) - 4\zeta^3(\lambda)}\Big)/2\right]^{1/3}, \quad (6)$$

where $\zeta(\lambda) = c_2^2(\lambda) - 3c_1(\lambda)c_3(\lambda)$, and $\eta(\lambda) = 2c_2^3(\lambda) - 9c_1(\lambda)c_2(\lambda)c_3(\lambda) + 27c_0(\lambda)c_3^2(\lambda)\lambda$. The asymptotic PDF of λ is then given by

$$\widetilde{p}(\lambda) = \frac{1}{2\sqrt{3}\pi\lambda c_3(\lambda)}\left(G(\lambda) - \frac{\zeta(\lambda)}{G(\lambda)}\right)\mathbb{1}_{[\lambda_-,\lambda_+]}. \quad (7)$$

Here $\mathbb{1}_{[\lambda_-,\lambda_+]}$ is the indicator function, meaning that $\widetilde{p}(\lambda)$ is nonzero only in $[\lambda_-, \lambda_+]$. The extremes λ_- and λ_+ are determined from the two real roots of the quartic equation given by $[\eta^2(\lambda) - 4\zeta^3(\lambda)]/[\lambda^2(b\beta\lambda + a\alpha)^2] = 0$. Notice that $\eta^2(\lambda) - 4\zeta^3(\lambda)$ is an 8-th degree polynomial in λ, but possesses λ^2 and $(b\beta\lambda + a\alpha)^2$ as factors, and hence the division results in a quartic polynomial. The cube root in (6) should be supplied the real solution if its argument happens to be negative, e.g. $(-8)^{1/3} = -2$, and not one of the two other complex solutions. We also observe that outside $[\lambda_-, \lambda_+]$, $\eta^2(\lambda) - 4\zeta^3(\lambda)$ becomes negative, thereby rendering its square-root in (6) as complex. It is possible to obtain the roots explicitly using Ferrari's method. However, it is quite lengthy, owing to the complicated coefficients in the quartic equation. Therefore, it is much more convenient to simply locate the extremes by solving the quartic equation numerically. Moreover, the extremes can be also inferred by evaluating the asymptotic density without the indicator function part, and figuring out the interval outside of which it returns complex values.

We emphasize that (7) holds even when $\mathbf{W}_A$ and $\mathbf{W}_B$ are real Wishart matrices from the distributions $\mathcal{W}_n(n_A, \mathbf{I}_n)$ and $\mathcal{W}_n(n_B, \mathbf{I}_n)$, respectively. Furthermore, (7) includes the Marčenko-Pastur density [1] as a special case when $a = 1$ and $b = 0$. To the best of our knowledge, expression (7) is novel.

In Fig. 1 we consider three sets of values for the parameters n, n_A, n_B, a, b, and plot the resulting eigenvalue densities. The empirical PDFs obtained from numerical simulations is contrasted with asymptotic densities, and a very good agreement is observed. For $n = 5$ and $n = 8$ we also show exact densities based on (2). However, for $n = 50$ evaluation using (2) is not feasible.

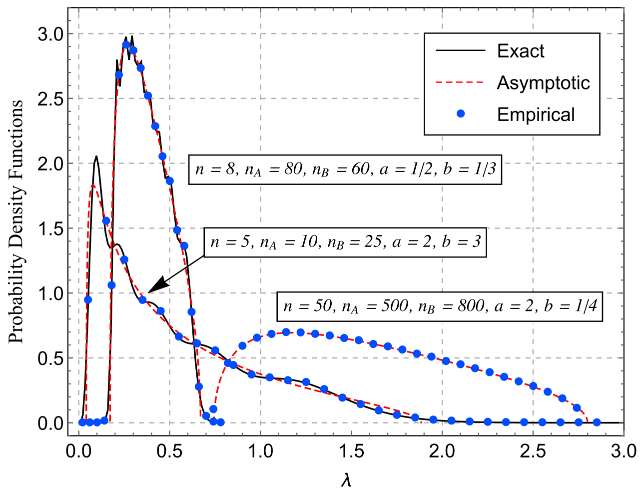

Fig. 1. Exact, asymptotic, and empirical eigenvalue densities of $\mathbf{Q}$.

III. Application to the Analysis of a Multi-Hop MIMO Relay System

A. System Model

The scenario under consideration is shown in Fig. 2, where the source (S) wants to communicate with the destination (D) with the help of K relays (R_k), $k = 1, \ldots, K$. The communication protocol used by the relays is the decode-and-forward. Also, end-to-end communication always consists of multi-hops, i.e., direct communication is not possible.

We assume that source and the relays have M transmitting antennas, and the destination and the relays have N receiving antennas. Self-interference may occur when the relay is operating in full-duplex mode, since it transmits and receives simultaneously. The complex discrete-time received signal $\mathbf{y}_k \in \mathbb{C}^{N\times 1}$ at k-th relay can be written as

$$\mathbf{y}_k = \mathbf{H}_k\sqrt{\rho_{k-1}}\mathbf{x}_{k-1} + \bar{\mathbf{H}}_k\sqrt{\eta_k}\mathbf{x}_k + \mathbf{z}_k, \tag{8}$$

where $\mathbf{x}_{k-1} \in \mathbb{C}^{M\times 1}$ and $\mathbf{x}_k \in \mathbb{C}^{M\times 1}$ are transmitted signals from nodes $k-1$ and k, respectively. Note that for $k = 1$, $k-1 = 0$ corresponds to the source node. The received signal at the destination is given by $\mathbf{y}_D = \mathbf{H}_{K+1}\sqrt{\rho_K}\mathbf{x}_K + \mathbf{z}_D$.

The power constraint on the transmitted signal is given by $\mathbb{E}[\mathbf{x}_k^{\dagger}\mathbf{x}_k] \leqslant M$, where $\mathbb{E}[\cdot]$ denotes the expectation operator and $\dagger$ stands for the conjugate transpose. The parameters $\rho_k = \eta_k = \mathrm{SNR}_k/M$ are the average received SNR at each receive antenna [16]. The variables $\mathbf{z}_k$ and $\mathbf{z}_D$ are the normalized additive white Gaussian noise at the relay k and destination, respectively, uncorrelated with $\mathbf{x}_k$ and $\mathbf{x}_{k-1}$. The matrix $\mathbf{H}_k \in \mathbb{C}^{N\times M}$ represents the channel gains, as depicted in Fig. 2. We consider flat and spatially uncorrelated Rayleigh fading MIMO channels. The entries of each matrix are independent and identically distributed (i.i.d.) complex Gaussian variables, with zero-mean and unit variance. We assume that the $\mathbf{H}_k$ is known at the receiver node only. The matrix $\bar{\mathbf{H}}_k \in \mathbb{C}^{N\times M}$ has the same distribution [17] as $\mathbf{H}_k$, and represents the channel between the transmitter and receiver antennas at the k-node, i.e., the self-interference path.

Define $\mathbf{W}_k$ as $\mathbf{H}_k\mathbf{H}_k^{\dagger}$ if $N \leqslant M$, and $\mathbf{H}_k^{\dagger}\mathbf{H}_k$ if $N > M$. The resulting random matrix $\mathbf{W}_k$ is non-negative definite, and thus has real, non-negative eigenvalues. It belongs to the ensemble of uncorrelated central complex Wishart matrices. A similar definition holds for $\bar{\mathbf{W}}_k$. Note that $\mathbf{W}_k$ and $\bar{\mathbf{W}}_k$ are $n \times n$–dimensional, where $n = \min(N, M)$.

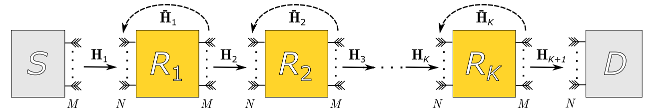

Fig. 2. Full-duplex multi-hop MIMO relay network. The solid lines show the desired signal, and the dashed lines show the self-interference.

B. Mutual Information

We assume for simplicity that the relays do not employ any active self-interference cancellation at the transmitters. Due to the self-interference, a MIMO MAC situation can be assumed for each relay, with the interpretation that there are two transmitters and one receiver. A similar model was used in [6] to account for two users wishing to communicate with a receiver using a given NOMA scheme.

One transmission occurs from node $k-1$ to k and other from k to k (self-interference), as can be observed in (8). With this assumption, the following rates can be achieved [9]: $\mathcal{I}_k + \bar{\mathcal{I}}_k \leqslant \log_2\left[\det\left(\mathbf{I}_n + \rho_{k-1}\mathbf{W}_k + \eta_k\bar{\mathbf{W}}_k\right)\right]$ and $\bar{\mathcal{I}}_k \leqslant \log_2\left[\det\left(\mathbf{I}_n + \eta_k\bar{\mathbf{W}}_k\right)\right]$. Here $\bar{\mathcal{I}}_k$ denotes the mutual information of self-interference, and $\mathcal{I}_k$ denotes the mutual information between transmitter $k-1$ and receiver k, measured in bits/channel-use.

Using the properties of logarithm, and employing the results $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$ and $\det(\mathbf{A}^{-1}) = \det(\mathbf{A})^{-1}$, where $\mathbf{A}$ and $\mathbf{B}$ are square matrices of same dimension, and assuming that the network is operating in one of the corner points of the MIMO MAC capacity region [9], we have

$$\mathcal{I}_k = \log_2\left[\det\left(\mathbf{I}_n + \frac{\rho_{k-1}\mathbf{W}_k}{\mathbf{I}_n + \eta_k\bar{\mathbf{W}}_k}\right)\right] = \log_2\left[\det\left(\mathbf{I}_n + \mathbf{Q}\right)\right], \tag{9}$$

where $\mathbf{Q}$ is as in (1). Moreover, the parameters a, b, n_A and n_B are contained in the relations $\rho_{k-1} = (a/n_A)$ and $\eta_k = (b/n_B)$. For the last hop, the mutual information is the same as in a MIMO single-hop, given by $\mathcal{I}_D = \log_2\left[\det\left(\mathbf{I}_n + \rho_K\mathbf{W}_{K+1}\right)\right]$ [9, eq. (8)].

C. Ergodic Capacity Upper Bound

Ergodic capacity is an appropriate capacity metric for channels that vary quickly, and therefore the channel is ergodic over the duration of one codeword [9]. Considering the mutual information for one hop with self-interference $\mathcal{I}_k$ given in (9), the ergodic capacity is given by [18]

$$C_k = \mathbb{E}\left\{\mathcal{I}_k\right\} \approx n\int_{\lambda_-}^{\lambda_+}\log_2(1+\lambda)\,\widetilde{p}(\lambda)\;d\lambda, \tag{10}$$

where $\widetilde{p}(\lambda)$ is given by (7), and the average $\mathbb{E}\{\cdot\}$ is taken over the ensemble of random matrices $\mathbf{Q}$. The ergodic capacity for the last hop C_D can be evaluated by setting $\eta_k = 0$ (i.e. $b = 0$).

Under the assumption that the hops are subject to independent fading due to the geographical distance between

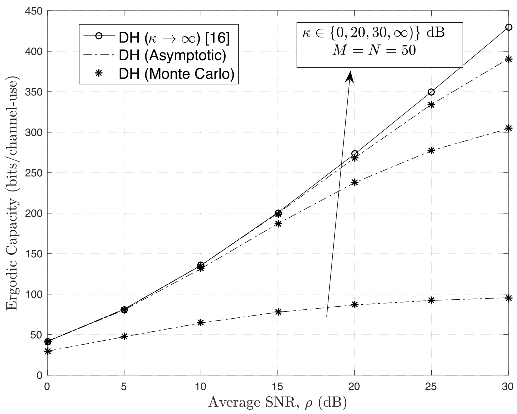

Fig. 3. Ergodic capacity for a dual-hop scenario.

TABLE II

PERCENTAGE GAP FOR A DUAL HOP SCENARIO ($\kappa = 30$ dB)

	Δ for $\rho = 5$	Δ for $\rho = 25$
$n = 5$, $n_A = 10$, $n_B = 25$, $a = 2$ and $b = 3$	4.70%	4.68%
$n = 8$, $n_A = 80$, $n_B = 60$, $a = 1/2$ and $b = 1/3$	1.21%	1.20%
$n = 50$, $n_A = 500$, $n_B = 800$, $a = 2$ and $b = 1/4$	0.15%	0.1488%

the nodes, according to the min-cut max-flow theorem [19], the overall multi-hop ergodic capacity C_{MH} is defined using the capacity of each individual link as $C_{\mathrm{MH}} = \mathbb{E}\{\min[\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_K, \mathcal{I}_D]\}$. The evaluation of C_{MH} using this expression would require solving a multi-dimensional integral. To circumvent this complication and take advantage of the ergodic capacity result for each hop given in (10), we use the Jensen's inequality and propose the following upper bound [20, eq. (20)] in the asymptotic limit of large number of antennas,

$$\begin{aligned} C_{\mathrm{MH}} &= \mathbb{E}\{\min[\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_K, \mathcal{I}_D]\} && (11)\\ &\leq \min\{\mathbb{E}[\mathcal{I}_1], \mathbb{E}[\mathcal{I}_2], \ldots, \mathbb{E}[\mathcal{I}_K], \mathbb{E}[\mathcal{I}_D]\} \\ &= \min\{C_1, C_2, \ldots, C_K, C_D\} = C_{\mathrm{MH}}^U, && (12) \end{aligned}$$

where C_k, $k = 1, \ldots, K$, and C_D are obtained using (10).

IV. NUMERICAL RESULTS AND REMARKS

We present here some case studies of multi-hop networks to show the application of the ergodic capacity expression derived in Section III. We demonstrate, with the aid of Monte Carlo simulations, that the proposed upper bound (12) is very tight with respect to (11) for all cases investigated. Fig. 3 depicts some plots of the ergodic capacity for a dual-hop (DH) scenario with four different attenuation levels of self-interference ($\eta_1 = \rho_0 - \kappa$), where $\kappa = 0, 20, 30$ dB for $M = N = 50$. For simplicity, all cases consider symmetrical multi-hop networks, i.e., $\rho_k = \rho_D = \rho$, for $k = 1, \ldots, K$. It can be observed that the ergodic capacity upper bound with the asymptotic expressions is in close agreement with the numerically simulated results. Moreover, the ergodic capacity increases as the self-interference attenuation increases, tending to the DH with no self-interference case, $\kappa \to \infty$, investigated in [21]. Additionally, it can be concluded that the ergodic capacity upper bound describes with great accuracy the numerical results even for small number of antennas. This can be seen in Table II, by means of the percentage gap $\Delta = 100 \cdot (C_{\mathrm{MH}}^U - C_{\mathrm{MH}})/C_{\mathrm{MH}}$.

V. CONCLUSIONS

We derived a novel expression for the asymptotic eigenvalue density of the quotient ensemble of Wishart matrices and applied it to obtain an upper bound on the ergodic capacity in a full-duplex multi-hop MIMO relay network. The results can also provide helpful insights into techniques such as NOMA, full-duplex, MIMO and cooperative systems.

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