Generation of Correlated Nakagami-m Variates with a Generalized Cross-Correlation

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***Abstract*— This paper presents a simple and efficient method for generation of Nakagami-m random variables taking cross-correlation with *arbitrary* correlation parameters between the Rayleigh in phase and quadrature components that compose the Nakagami-m envelope. It is applied the Cholesky decomposition method to correlate the samples with the desired correlation values. As an application, a simulation demonstrates the use of this method in the Outage Probability curve in a selection combiner receiver output operating in a Nakagami-m fading channel, considering different fading and correlation parameters between the signals.**

***Index Terms*— Cholesky, Cross-Correlation, Nakagami-m, Outage Probability**

I. Introduction

Several distributions are used in order to model the behavior of the signal in a fading environment. As an example, the Rayleigh distribution [1], which considers a non-line-of-sight (NLOS) propagation environment between transmitter and receiver, formed by Gaussian with zero mean and equal variances. The Rice distribution [2], which differs from the previous one by the fact of considering a line-of-sight (LOS) propagation environment, is composed by Gaussian with arbitrary non-zero means and equal variances. The Hoyt density (Nakagami-q) [3], which also considers a NLOS environment is formed by Gaussian with zero mean and arbitrary variance. The distributions aforementioned consider a propagation environment with only one cluster of multipath waves, however, considering that the received signal can be composed by multiple clusters, other broader distributions can be used, like Nakagami-m [4], α-μ [5], κ-μ [6] and η-μ [6], among others. The Nakagami-m distribution is widely used, both in practice and in theory for wireless communication fading channels, and its random variables generation is a key issue for simulations and test of such systems. A physical model for the Nakagami-m signal was proposed in [7]. In this model the Nakagami-m random variable can be obtained as the square root of the sum of squares of $2m$ independent Gaussian processes or, similarly, the square root of the sum of squares of m Rayleigh processes.

In order to simplify calculations and expressions, most studies consider that the received signals on different diversity branches are uncorrelated and can thus be separated and handled as independent signals [8]. However, this assumption of statistical independence between the various diversity branches is valid only if they are sufficiently separated [9] if we consider, for example, spatial diversity. However in wireless communications systems this separation is not always possible, and in order to improve the simulations accuracy, we should take into account the effect of this correlation has to generate random variables according to a given distribution.

In [10]–[15], there are a great number of proposed methods for generating Nakagami-m samples with arbitrary correlation parameters and/or arbitrary fading parameters but identical for both signals through the sum of variables Gamma. The authors of [16] show a method of adding correlation both temporal and spatial in Nakagami-m samples already generated to model MIMO fading channels, whereas [17] is a study of the effective capacity of this channel using such samples. In [18] is provided an algorithm for generation of Nakagami-m samples with arbitrary correlation parameters and fading and not necessarily identical making use of the Rice generalized distribution, with the limitation only to be applied in receiver with two diversity branches. In [19], exact mathematical expressions for Nakagami-m processes with two variables considering cross-correlation and arbitrary correlation and fading parameters are derived, being drawn a curve outage probability (OP) to a selection combiner (SC) receiver.

This paper aims at practical confirmation of the equation (21) of [19] through a very simple method of generating Nakagami-m correlated variables considering cross-correlation between the Rayleigh signals (in phase and quadrature components), with not necessarily identical fading parameters unlike most of the previously proposed methods and also with the advantage that it can be applied in as many receiving branches are needed.

The remainder of the paper is organized as follows. Section II presents the system model adopted in this paper. Section III describes presents simulation results and discussions concerning the influence of the system parameters. Finally, Section IV concludes the paper.

II. Model

A. The Nakagami-m Distribution

The Nakagami-m probability density function (PDF) is given by

$$f_R(r) = \frac{2m^m r^{2m-1}}{\Omega^m \Gamma(m)} \exp\left(-\frac{mr^2}{\Omega}\right) \tag{1}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) \mathrm{d}t \tag{2}$$

is the Gamma function, $\Omega = \mathbb{E}(R^2)$ is the average power of the Nakagami-m signal, with $\mathbb{E}(\cdot)$ the statistical expectation operator, m is the fading parameter representing the inverse of the normalized variance of R^2, expressed as

$$m = \frac{\Omega^2}{\mathbb{V}(R^2)} \tag{3}$$

and $\mathbb{V}(\cdot)$ is the variance operator. The parameter m is useful to describe the fading degree experienced by the signal propagating in a multipath environment. Thus, if $m < 1$ the fading is more severe than the Rayleigh model ($m = 1$). If $m > 1$ the fading is less severe than Rayleigh, and $m \to \infty$, reproduces a non-fading environment. The cumulative distribution function (CDF) of the Nakagami-m envelope is given by

$$F_R(r) = \frac{\gamma(m, mr^2/\Omega)}{\Gamma(m)} \tag{4}$$

where $\gamma(a, x)$ is the incomplete Gamma function expressed by [20]

$$\gamma(a, x) = \int_0^x \exp(-t) t^{a-1} \mathrm{d}t. \tag{5}$$

Rayleigh and Semi-Gausssian distributions are particular cases of the Nakagami-m distribution when we adjust $m = 1$ and $m = 0.5$, respectively.

B. Cross-Correlation Model Adopted

There are several models of correlation usually adopted. We can cite the constant model which can be obtained by antennas very close [21]; exponential model [22], which can be obtained through equidistant antennas; the circular model, which can be applied to antennas arranged in a circle or four antennas arranged in a square [21], and an arbitrary model (focus of this article) where a correlation is adopted arbitrarily for each pair of signal therefore wider than the models described above and with the advantage of being individualized in each case.

We define $Y = [\mathbf{Y}_i \ \mathbf{Y}_j]^T$, with $\mathbf{Y}_i$ and $\mathbf{Y}_j$ and column vectors with dimension $2m_1$ e $2m_2$, respectively, with $1 \leq i \leq 2m_1$ e $2m_1+1 \leq j \leq 2m_1+2m_2$. Without loss of generality, we can consider $m_2 \geq m_1$. The variables Y_k are distributed according to a Gaussian with zero mean and correlation matrix $\mathbf{\Sigma}$, whose elements are given by $\Sigma_{ij} \equiv 1$ to $i = j$ e $\Sigma_{ij} \equiv \rho_{ij}$ para $i \neq j$, where ρ_{ij} satisfies the conditions $-1 \leq \rho_{ij} \leq 1$ and $\rho_{ij} = \rho_{ji}$, such that

$$\rho_{k\ell} = \frac{\mathbb{C}(Y_k, Y_\ell)}{\sqrt{\mathbb{V}(Y_k, Y_\ell)}} \tag{6}$$

where $1 \leq k \leq 2m_1 + 2m_2$, $1 \leq \ell \leq 2m_1 + 2m_2$ and $\mathbb{C}(\cdot, \cdot)$ denoting the covariance operator.

A Nakagami-m variate can be obtained as the square root of a Gamma variate, which, in turn, is given by the sum of $2m$ squared independent Gaussian variables. In order to form the Nakagami process, set Y_k as the k-th in-phase component of the fading signal, for k odd, and Y_k as the k-th quadrature component of the fading signal, for k even. Now, allowing for arbitrary correlation coefficients between the in-phase and quadrature components of the Rayleigh processes composing the Nakagami signals, then

$$\rho_{k\ell} = \begin{cases} \delta_1, & k \text{ odd and } \ \ell = 2m_1 + k, \quad\quad k < 2m_1 \\ \delta_2, & k \text{ even and } \ \ell = 2m_1 + k, \quad\quad k \leq 2m_1 \\ \delta_3, & k \text{ odd and } \ \ell = 2m_1 + k + 1, \ k < 2m_1 \\ \delta_4, & k \text{ even and } \ \ell = 2m_1 + k - 1, \ k \leq 2m_1 \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

for which $\rho_{k\ell} = \rho_{\ell k}$. Figure 1 shows, in a pictorial perspective, the arbitrary correlation pattern defined before, where R_1 and R_2 are Nakagami distributed envelopes. This figure establishes that the correlation between the pairs of Rayleigh processes composing the two Nakagami signals occurs as follows:

- δ_1 is the correlation between the in-phase components;
- δ_2 is the correlation between the quadrature components;
- δ_3 is the crosscorrelation between in-phase and quadrature components and
- δ_4 is the crosscorrelation between quadrature and in-phase components.

This paper adopts the model of cross-correlation previously mentioned, proposed in [19] as a way to correlate the Nakagami-m samples generated.

C. Cholesky Decomposition Method

The Cholesky decomposition method decomposes the correlation matrix $\mathbf{\Sigma}$ in another matrix $\mathbf{L}$, which multiplied by its transposed hermitian matrix, has resulted the same matrix $\mathbf{\Sigma}$, and

$$\mathbf{\Sigma} = \mathbf{L}\mathbf{L}^\dagger, \tag{8}$$

where $\mathbf{L}^\dagger$ denotes the transposed hermitian matrix of $\mathbf{L}$. We now generate a Gaussian independent identically distributed (i.i.d.) sequence with zero mean and unit variance, that is

$$\mathbf{e} = \mathcal{N}(0, 1). \tag{9}$$

Then the vector

$$\mathbf{x} = \mathbf{L}\mathbf{e} \tag{10}$$

will be Gaussian distribution correlated according to correlation matrix $\mathbf{\Sigma}$.

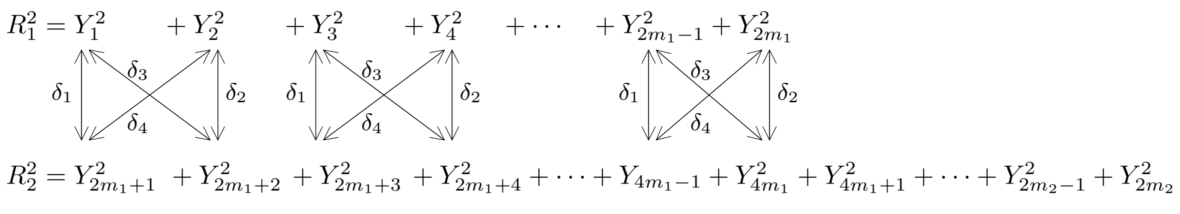

Fig. 1. The arbitrary correlation defined in [19] and equation (7) in a pictorial perspective.

D. Methodology

The methodology described below is applied to the case of only two receiving branches but can be adapted for use in cases with any number of branches in reception:

1) Get $\mathbf{\Sigma}$, a square matrix $2m_1 + 2m_2$ containing the correlation coefficients described in (7);
2) Apply at matrix $\mathbf{\Sigma}$ the Cholesky decomposition method, resulting in a lower triangular matrix $\mathbf{L}$, with same size as $\mathbf{\Sigma}$;
3) Generate a matrix $\mathbf{N}$ of size $(2m_1 + 2m_2) \times n$, where n is the number of samples, as follows:
   a) Line 1 to line $2m_1$, the matrix's rows are i.i.d. Gaussian sequences with zero mean and variance $1/2m_1$, as described in (3);
   b) Line $2m_1 + 1$ to line $2m_2$, the matrix's rows are the same as above, but with variance $1/2m_2$.
4) Do $\mathbf{M} = \mathbf{LN}$;
5) Get $R_1 = \sum_{i=1}^{2m_1} M_i^2$ e $R_2 = \sum_{i=2m_1+1}^{2m_2} M_i^2$, where M_i indicates the i-th row of the matrix $\mathbf{M}$;
6) R_1 and R_2 are Nakagami-m samples with their respective values of m (m_1 and m_2, respectively), and correlated according to the correlation matrix $\mathbf{\Sigma}$.

Importantly, the method described here can only be applied to integer values of $2m_1$ and $2m_2$, i.e., the fading parameters m_1 and m_2 can be half integer or integer values in the proposed method.

III. Application: Outage Probability

The signal-to-noise ratio (SNR) obtained in the i-th reception branch is given by

$$\Gamma_i = R_i^2 E_b / N_0 \tag{11}$$

where E_b/N_0 is the ratio of the energy per bit and the noise power spectrum and $\overline{\gamma}_i = E_b/N_0 \mathbb{E}(R_i^2)$ is the average signal-to-noise ratio. The SC receiver chooses the branch with the maximum SNR instantaneous γ for a symbol decision. Then the output SNR is equal to $\Gamma = \max\{\Gamma_1, \Gamma_2\}$ [21]. The equation (12) is the reproduction of the expression (21) of [19], and provides the CDF of γ in a SC receiver output.

A outage probability curve describes the probability of a receiver to be out of operation (reliability) as a function of its SNR input. Figure 2 shows the Outage Probability curve in a SC receiver output. In it, the points represent the simulation values, while the solid lines represent the theoretical values obtained with (12). We used 10^8 points for simulation. Note the excellent adherence of sample values with the theoretical values, which gives excellent efficacy to the proposed method. It was used to simulate $\overline{\gamma}_i = 1$.

Two different sets of curves were made with two pairs of fading parameters, one with $m_1 = 1.5$ and $m_2 = 2$, and other with $m_1 = 2.5$ and $m_2 = 3$. Since for $m \to \infty$ reproduces a non-fading environment, one would expect better performance for the receiver with the highest values of m, which can be confirmed in the figure. The correlation values δ_1, δ_2, δ_3 and δ_4 used are described in the figure, and the modify of any of these four parameters leads a different performance. When considering correlation between all components that form the Nakagami signal, we have the worst performance for the receiver in both cases, while the best performance in both cases is obtained with values $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 0$, i.e., considering a non-correlated environment. Intermediate situations are found when we consider $\delta_1 = \delta_2$ and $\delta_3 = -\delta_4$, and when we do not consider cross correlations between phase and quadrature components with $\delta_3 = \delta_4 = 0$ and $\delta_1 \neq 0$ and $\delta_2 \neq 0$.

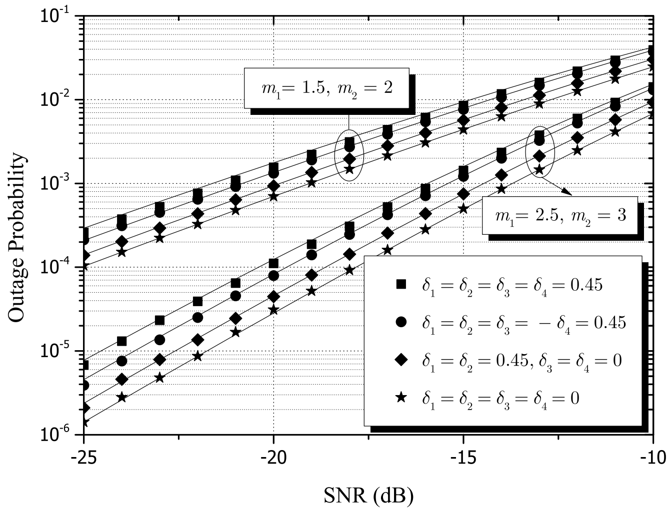

Fig. 2. Outage probability in a SC receiver output working in a Nakagami-m correlated fading channel.

IV. Conclusion

This paper proposes the use of a simple and effective method for the generation of Nakagami-m random variables, considering cross-correlation between the components that form the signal. The fact that algorithm accept different values of m for each diversity branch, and, additionally, a number of branches greater than two in reception, is an important differential that should be taken into consideration. The outage probability curve generated exemplifies the usefulness and effectiveness of the proposed method, that it can be used to simulate various diversity and multiplexing systems, as Selection Combiners, Equal-Gain Combiners and Maximal-Ratio Combiners, as in MIMO systems operating in Nakagami-m channels.

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\begin{aligned}
F_\Gamma(\gamma) = &\sum_{r=0}^{\infty} \frac{(m_1/2)_r}{r!} \sum_{k=0}^{r} \binom{r}{k} (-1)^k (\delta_1\delta_2 - \delta_3\delta_4)^{2k} (\delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2)^{r-k} \\
&\times \sum_{j=0}^{r+k} \binom{r+k}{j} \frac{(-1)^j \gamma\left(m_1 + j, m_1 \frac{\gamma}{\overline{\gamma}_1}\right)}{\Gamma(m_1 + j)} \sum_{i=0}^{r+k} \binom{r+k}{i} \frac{(-1)^i \gamma\left(m_2 + i, m_2 \frac{\gamma}{\overline{\gamma}_2}\right)}{\Gamma(m_2 + i)}
\end{aligned} \tag{12}
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